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# Triangle inequality for quantum integral operator

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**Abstract**

We show that general integral triangle inequality does not hold for shifted $q$-integrals. Furthermore, we obtain a triangle inequality for shifted $q$-integrals. We also give an estimate for $q$-integrable product and use it to refine some recently obtained Ostrowski inequalities for quantum calculus.

*Keywords:* q-derivative, q-integral, integral triangle inequality, Ostrowski inequality
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## 1 Introduction

Quantum calculus is a calculus that does not use the concept of the limit, as it is based on finite differences. Herein, we will consider a branch of quantum calculus called $q$-calculus. Basic notions in this type of calculus, $q$-derivative and $q$-integral, were introduced by F.H.Jackson [4]. In this work we focus our attention on shifted $q$-derivative and $q$-integral.

This manuscript is organized as follows, in Section 2 we give $q$-calculus preliminaries, we state basic definitions and properties for shifted $q$-derivatives and shifted $q$-integrals known from the literature [10]. Afterwards, in Section 3 we show by counterexamples that for $f : [a, x] \to \mathbb{R}$ that is $q$-integrable, the triangle inequality in general does not hold for every $c \in [a, x]$:

$$\left| \int_c^x f(t) d_q^a t \right| \nleq \int_c^x |f(t)| \, d_q^a t \quad (a \leq c \leq x < +\infty). \tag{1}$$

However, when $c$ is on the $q$-lattice, which means that it is of the form $c = a + q^m(x - a)$, fore some $m \in \mathbb{N}_0$, we show that triangle inequality holds. In Section 4 we give an estimate for the $q$-integrable product, also valid only when the lower bound of the $q$-integral is on the $q$-lattice. We will use this estimate, together with obtained triangle inequality from Section 3 to finally show, in Section 5, refinements of recently obtained Ostrowski inequalities (see [8]).

## 2 Shifted $q$-derivative and $q$-integral

Let $q \in \langle 0, 1 \rangle$. The **shifted $q$-derivative** of an arbitrary function $f$ : $[a, b] \to \mathbb{R}$ is defined by (see [10])

$$D_q^a f(x) = \frac{f(x) - f(a + q(x - a))}{(1 - q)(x - a)}, \quad x \in \langle a, b], \\ D_q^a f(a) = \lim_{x \to a} D_q^a f(x).$$

Note that every such function is $q$-differentiable for every $x \in \langle a, b]$ and, if $\lim_{x \to a} D_q^a f(x)$ exists, it is $q$-differentiable on $[a, b]$. The shifted $q$-derivative is a generalization of the **Euler-Jackson $q$-difference operator** (see [4]) and both are discretizations of ordinary derivative, and if $f$ is differentiable function then

$$\lim_{q \to 1} D_q^a f(x) = f'(x).$$

Shifted $q$-integral (a generalization of Jackson integral) is defined by

$$\int_a^x f(t) \, d_q^a t = (1 - q)(x - a) \sum_{k=0}^{\infty} q^k f\left(a + q^k (x - a)\right), \quad x \in [a, b]. \tag{2}$$

If the series on the right hand-side converges, then $q$-integral $\int_a^x f(t) d_q^a t$ exists. If $f$ is continuous function on $[a, b]$, the series

$$(1 - q)(x - a) \sum_{k=0}^{\infty} q^k f\left(a + q^k (x - a)\right)$$

tends to the Riemann integral when $q \to 1$ ([3], [6]):

$$\lim_{q\to 1} \int_a^x f(t)\, d_q^a t = \int_a^x f(t)\, dt.$$

The shifted $q$-integral is a generalization of the **Jackson $q$-integral** (see [5]). If $c \in \langle a, x\rangle$ shifted $q$-integral is defined by

$$\int_c^x f(t)\, d_q^a t = \int_a^x f(t)\, d_q^a t - \int_a^c f(t)\, d_q^a t. \tag{3}$$

In the following theorem important properties of shifted $q$-derivatives and $q$-integrals are given (see [10]).

**Theorem 1.** *For a function $f\colon [a,b] \to \mathbb{R}$, $q \in \langle 0,1\rangle$ and $x \in [a,b]$, the following identities hold:*

*(i)*

$$D_q^a \left( \int_a^x f(t)\, d_q^a t \right) = f(x),$$

*(ii)*

$$\int_a^x D_q^a f(t)\, d_q^a t = f(x) - f(a)$$

*(iii)*

$$\int_a^x (f(t) + g(t))\, d_q^a\, t = \int_a^x f(t)\, d_q^a t + \int_a^x g(t)\, d_q^a t$$

*(iv)*

$$\int_a^x \alpha f(t)\, d_q^a t = \alpha \int_a^x f(t)\, d_q^a\, t, \alpha \in \mathbb{R}.$$

In the next section we will show that general integral triangle inequality does not hold for shifted $q$-integrals, as stated in (1). Furthermore, we will show that triangle inequality for shifted $q$-integrals is valid when lower integral bound is a point of the $q$-lattice.

# 3 Triangle inequality for shifted $q$-integrals

In [3], (Section 1.3.1, Remark (ii)) an example of a function is given for which the triangle inequality for $q$-integral (Jackson integral) does not hold. Following this example, here we give an example of a function for which the triangle inequality for shifted $q$-integrals does not hold.

**Example 1.** *Let us consider the function* $f\colon [a,b] \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} \frac{4q^{-n}(x-a)-(1+3q)(b-a)}{(1-q)(b-a)^2}, & a+q^{n+1}(b-a) \le x \le a + \frac{q^n(q+1)}{2}(b-a), \\ \frac{4\left(-q^{-n}(x-a)+(b-a)\right)}{(1-q)(b-a)^2}, & a+\frac{q^n(q+1)}{2}(b-a) < x \le a+q^n(b-a), \\ 0, & x=a. \end{cases}$$

*where* $n \in \mathbb{N}_0$*. This function is visualized in Figure 1.*

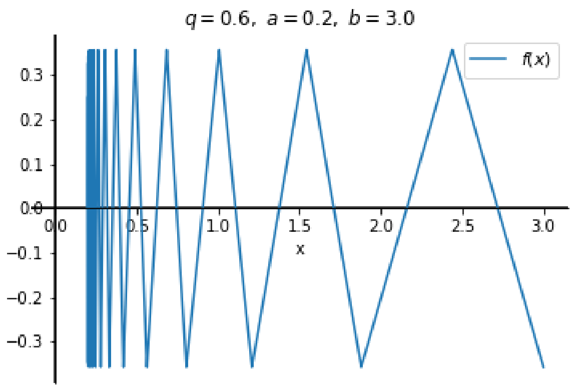


Figure 1: The function $f(x)$ on $[0.2, 3.0]$ when $q = 0.6$

*It easily follows that the function* $f$ *is continuous. Furthermore, at the points of* $q$*-lattice it attains the value* $-\frac{1}{b-a}$*, while in the midpoints of* $q$*-lattice it attains the value* $\frac{1}{b-a}$*:*

$$f\Big(a+q^n(b-a)\Big) = -\frac{1}{b-a}, \quad n \in \mathbb{N},$$

*and*

$$f\left(a+\frac{q^n(q+1)}{2}(b-a)\right) = \frac{1}{b-a}, \quad n \in \mathbb{N}.$$

*In order to show that triangle inequality is not valid we calculate the following* $q$*-integral:*

$$\int_{a+\frac{q+1}{2}(b-a)}^{b} f(t)\, d_q^a\, t = \int_a^b f(t)\, d_q^a\, t - \int_a^{a+\frac{q+1}{2}(b-a)} f(t)\, d_q^a\, t$$

$$= (1-q)(b-a)\sum_{k=0}^{\infty} q^k f\left(a + q^k (b-a)\right)$$

$$-(1-q)(b-a)\frac{1}{2}(q+1)\sum_{k=0}^{\infty} q^k f\left(a + \frac{q^k (q+1)}{2}(b-a)\right)$$

$$= (1-q)(b-a)\left(\sum_{k=0}^{\infty} q^k \left(-\frac{1}{b-a}\right) - \frac{1}{2}(q+1)\sum_{k=0}^{\infty} q^k \left(\frac{1}{b-a}\right)\right)$$

$$= -1 - \frac{1}{2}(q+1) = -\frac{q+3}{2}$$

*and we obtain*

$$\left|\int_{a+\frac{q+1}{2}(b-a)}^{b} f(t)\, d_q^a\, t\right| = \frac{q+3}{2}.$$

*Now, we calculate*

$$\int_{a+\frac{q+1}{2}(b-a)}^{b} |f(t)|\, d_q^a\, t = \int_a^b |f(t)|\, d_q^a\, t - \int_a^{a+\frac{q+1}{2}(b-a)} |f(t)|\, d_q^a\, t$$

$$= (1-q)(b-a)\sum_{k=0}^{\infty} q^k \left|f\left(a + q^k (b-a)\right)\right|$$

$$-(1-q)(b-a)\frac{1}{2}(q+1)\sum_{k=0}^{\infty} q^k \left|f\left(a + \frac{q^k (q+1)}{2}(b-a)\right)\right|$$

$$= (1-q)(b-a)\left(\sum_{k=0}^{\infty} q^k \left(\frac{1}{b-a}\right) - \frac{1}{2}(q+1)\sum_{k=0}^{\infty} q^k \left(\frac{1}{b-a}\right)\right)$$

$$= 1 - \frac{1}{2}(q+1) = \frac{1-q}{2}.$$

*We have shown that*

$$\left|\int_{a+\frac{q+1}{2}(b-a)}^{b} f(t)\, d_q^a\, t\right| > \int_{a+\frac{q+1}{2}(b-a)}^{b} |f(t)|\, d_q^a\, t,$$

*and therefore the triangle inequality is not valid in general, for every* $c \in [a,b]$*, as written in* (1)*.*

**Example 2.** *Now we consider the function* $f:[a,b]\to\mathbb{R}$ *defined by*

$$f(x)=\frac{b-x}{b-a},\quad x\in[a,b]. \tag{4}$$

*Using the function*(4) *we calculate*

$$\begin{aligned}\int_{\frac{a+b}{2}}^{b} f(t)\,d_q^a\,t &= \int_a^b f(t)\,d_q^a\,t-\int_a^{\frac{a+b}{2}} f(t)\,d_q^a\,t\\ &=(1-q)(b-a)\left(\sum_{k=0}^{\infty}q^k f\left(a+q^k(b-a)\right)-\frac{1}{2}\sum_{k=0}^{\infty}q^k f\left(a+q^k\frac{b-a}{2}\right)\right)\\ &=(1-q)(b-a)\left(\sum_{k=0}^{\infty}q^k\left(1-q^k\right)-\frac{1}{2}\sum_{k=0}^{\infty}q^k\left(1-\frac{1}{2}q^k\right)\right)\\ &=(b-a)\left(\frac{q}{1+q}-\frac{1+2q}{4(1+q)}\right)=(b-a)\frac{2q-1}{4(1+q)}.\end{aligned}$$

*For every* $q\in\left\langle 0,\frac{1}{2}\right\rangle$ *we have*

$$\left|\int_{\frac{a+b}{2}}^{b} f(t)\,d_q^a\,t\right|=(b-a)\frac{1-2q}{4(1+q)}.$$

*Let us note that the function* (4) *is positive, so it follows that*

$$\int_{\frac{a+b}{2}}^{b}|f(t)|\,d_q^a\,t=\int_{\frac{a+b}{2}}^{b} f(t)\,d_q^a\,t=(b-a)\frac{2q-1}{4(1+q)}<0.$$

*We have shown once more that the triangle inequality for shifted* $q$*-integrals generally does not hold as we have obtained*

$$\left|\int_{\frac{a+b}{2}}^{b} f(t)\,d_q^a\,t\right|=(b-a)\frac{1-2q}{4(1+q)}>-(b-a)\frac{1-2q}{4(1+q)}=\int_{\frac{a+b}{2}}^{b}|f(t)|\,d_q^a\,t.$$

Now, we will show that the triangle inequality is valid when lower bound of the $q$-integral is on the $q$-lattice. In the next lemma the triangle inequality for shifted $q$-integrals is given.

**Lemma 1.** *Let $f : [a,b] \to \mathbb{R}$ be such that $|f|$ is a q-integrable function on $[a,b]$. Then for every $x \in \langle a,b]$ and $m \in \mathbb{N}_0$, the following inequality holds*

$$\left| \int_{a+q^m(x-a)}^{x} f(t)\, d_q^a\, t \right| \leq \int_{a+q^m(x-a)}^{x} |f(t)|\, d_q^a\, t, \tag{5}$$

*Proof.* From the definitions (2) and (3) we have

$$\begin{aligned}
\left| \int_{a+q^m(x-a)}^{x} f(t)\, d_q^a\, t \right| &= \left| \int_a^x f(t)\, d_q^a\, t - \int_a^{a+q^m(x-a)} f(t)\, d_q^a\, t \right| \\
&= \Bigg| (1-q)(x-a) \sum_{k=0}^{\infty} q^k f\left(a + q^k (x-a)\right) \\
&\quad - (1-q)(q^m (x-a)) \sum_{k=0}^{\infty} q^k f\left(a + q^{k+m}(x-a)\right) \Bigg| \\
&= \left| (1-q)(x-a) \sum_{k=0}^{m-1} q^k f\left(a + q^k (x-a)\right) \right|.
\end{aligned}$$

Now we use the discrete triangle inequality for finite sequence to obtain

$$\begin{aligned}
&\left| (1-q)(x-a) \sum_{k=0}^{m-1} q^k f\left(a + q^k (x-a)\right) \right| \\
&\leq (1-q)(x-a) \sum_{k=0}^{m-1} q^k \left| f\left(a + q^k (x-a)\right) \right| \\
&= (1-q)(x-a) \sum_{k=0}^{\infty} q^k \left| f\left(a + q^k (x-a)\right) \right| \\
&- (1-q)(q^m (x-a)) \sum_{k=0}^{\infty} q^k \left| f\left(a + q^{k+m}(x-a)\right) \right| \\
&= \int_a^x |f(t)|\, d_q^a\, t - \int_a^{a+q^m(x-a)} |f(t)|\, d_q^a\, t = \int_{a+q^m(x-a)}^{x} |f(t)|\, d_q^a\, t.
\end{aligned}$$

□

**Remark 1.** *If we take $m = 0$ in (5) we obtain that for every $x \in \langle a,b]$*

$$\left| \int_a^x f(t)\, d_q^a\, t \right| \leq \int_a^x |f(t)|\, d_q^a\, t.$$

In the following Section we give an estimate for q-integrable product. This estimate does not hold in general, but only for the lower bound of the $q$-integral that is on $q$-lattice, as we will see in Theorem 2.

# 4 Estimate for $q$-integrable product

Here and hereafter we suppose that $f\colon [a,b] \to \mathbb{R}$ is $q$-differentiable on $[a,b]$, therefore the limit $\lim\limits_{x\to a} D_q^a f(x)$ exists.

The symbol $L_q^\infty [a,b]$ denotes the space of bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty^{[a,b]} = \sup_{t\in[a,b]} |f(t)| .$$

**Theorem 2.** *Suppose that $f\colon [a,b] \to \mathbb{R}$ is such that $|f|$ is a q-integrable function on $[a,b]$. Furthermore, let $g : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$. Then, for every $x \in \langle a,b]$ and $m \in \mathbb{N}_0$ the following identity holds*

$$\int_{a+q^m(x-a)}^{x} |f(t)|\,|g(t)|\, d_q^a\, t \le \|g\|_\infty^{[a,x]} \int_{a+q^m(x-a)}^{x} |f(t)|\, d_q^a\, t. \tag{6}$$

*Proof.* From the definitions (2) and (3) we have

$$\int_{a+q^m(x-a)}^{x} |f(t)|\,|g(t)|\, d_q^a\, t = \int_a^x |f(t)|\,|g(t)|\, d_q^a\, t - \int_a^{a+q^m(x-a)} |f(t)|\,|g(t)|\, d_q^a\, t$$

$$= (1-q)(x-a)\sum_{k=0}^{\infty} q^k \left|f\left(a+q^k(x-a)\right)\right| \left|g\left(a+q^k(x-a)\right)\right|$$

$$-(1-q)\,q^m(x-a)\sum_{k=0}^{\infty} q^k \left|f\left(a+q^{k+m}(x-a)\right)\right| \left|g\left(a+q^{k+m}(x-a)\right)\right|$$

$$= (1-q)(x-a)\sum_{k=0}^{m-1} q^k \left|f\left(a+q^k(x-a)\right)\right| \left|g\left(a+q^k(x-a)\right)\right|$$

$$\le \|g\|_\infty^{[a,x]} (1-q)(x-a)\sum_{k=0}^{m-1} q^k \left|f\left(a+q^k(x-a)\right)\right|$$

$$= \|g\|_\infty^{[a,x]} (1-q)(x-a)\sum_{k=0}^{\infty} q^k \left|f\left(a+q^k(x-a)\right)\right|$$

$$- \|g\|_{\infty}^{[a,x]} (1-q) q^m (x-a) \sum_{k=0}^{\infty} q^k \left| f\left(a + q^{k+m}(x-a)\right)\right|$$

$$= \|g\|_{\infty}^{[a,x]} \int_{a+q^m(x-a)}^{x} |f(t)| \, d_q^a \, t.$$

□

**Remark 2.** *If we take $m=0$ in (6) we obtain that for every $x \in \langle a,b]$*

$$\int_a^x |f(t)| \, |g(t)| \, d_q^a \, t \leq \|g\|_{\infty}^{[a,x]} \int_a^x |f(t)| \, d_q^a \, t.$$

Now we will use the inequality (6) and (5) together with Montgomery identity stated in [1] to refine some recently obtained Ostrowski inequalities for shifted quantum integral operator.

# 5 Refinements of Ostrowski type inequalities for $q$-integrals

Ostrowski inequalities for which we will give refinements were proved by the mean value theorem for $q$-integrals in [2]. The first inequality is valid for every $x \in [a,b]$ and the second, with a tighter bound then the first, is valid only on the $q$-lattice, that is for $x = a + q^m(b-a)$, $m \in \mathbb{N}_0$. Next identity for $q$-integrals is given in [1].

**Lemma 2.** *Let $f : [a,b] \to \mathbb{R}$ be arbitrary function and $x \in [a,b]$. Then the following identity holds*

$$f(x) - \frac{1}{b-a} \int_a^b f(t) \, d_q^a t = (b-a) \int_0^{\frac{x-a}{b-a}} (qt) \, D_q^a f(tb + (1-t)a) \, d_q^0 t$$
$$+ (b-a) \int_{\frac{x-a}{b-a}}^{1} (qt-1) \, D_q^a f(tb + (1-t)a) \, d_q^0 t. \tag{7}$$

**Remark 3.** *In the case when $q=1$, identity (7) reduces to classic Montgomery identity for Riemann integral (see [7] or [9]).*

The following theorem gives generalization of **Ostrowski inequality and its refinement for $q$-integrals** that is valid for every $x \in [a,b]$.

**Theorem 3.** *(Refinement Ostrowski inequality for q-calculus) Let $f\colon [a,b] \to \mathbb{R}$ be a q-integrable function over $[a,b]$. For $x \in [a,b]$ following inequalities hold*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t \right| \leq \frac{(b-a)}{1+q} \left\| D_q^a f \right\|_\infty^{[a,b]} + (x-a) \left\| D_q^a f \right\|_\infty^{[a,x]}$$
$$\leq (b-a)\left(\frac{1}{1+q} + \frac{x-a}{b-a}\right) \left\| D_q^a f \right\|_\infty^{[a,b]}.$$

*Proof.* Starting from (7) we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a\, t \right| \\
&= (b-a)\left| \int_0^{\frac{x-a}{b-a}} (qt)\, D_q^a f(tb+(1-t)a)\, d_q^0 t + \int_{\frac{x-a}{b-a}}^1 (qt-1)\, D_q^a f(tb+(1-t)a)\, d_q^0 t \right| \\
&= (b-a)\left| \int_0^1 (qt-1)\, D_q^a f(tb+(1-t)a)\, d_q^0 t + \int_0^{\frac{x-a}{b-a}} D_q^a f(tb+(1-t)a)\, d_q^0 t \right| \\
&\leq (b-a)\left( \left| \int_0^1 (qt-1)\, D_q^a f(tb+(1-t)a)\, d_q^0 t \right| + \left| \int_0^{\frac{x-a}{b-a}} D_q^a f(tb+(1-t)a)\, d_q^0 t \right| \right).
\end{aligned}$$

By using triangle inequality for $q$-integrals (5), inequality (6), and

$$\left\| D_q^a f(tb+(1-t)a) \right\|_\infty^{\left[0,\frac{x-a}{b-a}\right]} = \left\| D_q^a f \right\|_\infty^{[a,x]}$$

we have

$$\begin{aligned}
&(b-a)\left( \left| \int_0^1 (qt-1)\, D_q^a f(tb+(1-t)a)\, d_q^0 t \right| + \left| \int_0^{\frac{x-a}{b-a}} D_q^a f(tb+(1-t)a)\, d_q^0 t \right| \right) \\
&\leq (b-a) \left\| D_q^a f \right\|_\infty^{[a,b]} \int_0^1 (1-qt)\, d_q^0 t + (b-a) \left\| D_q^a f \right\|_\infty^{[a,x]} \left(\frac{x-a}{b-a}\right) \\
&= (b-a)\left( \left(1 - \frac{q}{1+q}\right) \left\| D_q^a f \right\|_\infty^{[a,b]} + \frac{x-a}{b-a} \left\| D_q^a f \right\|_\infty^{[a,x]} \right)
\end{aligned}$$

and the first inequality is proved. The second follows immediately after applying $\left\| D_q^a f \right\|_\infty^{[a,x]} \leq \left\| D_q^a f \right\|_\infty^{[a,b]}$. □

In next theorem we give generalization of **Ostrowski inequality and its refinement for $q$-integrals**. This inequality is valid only on the $q$-lattice, that is for $x = a + q^m (b - a)$, $m \in \mathbb{N}_0$. In the proof we will use the result from [11]:

$$\int_a^x (t-a)^n \, d_q^a \, t = \left( \frac{1-q}{1-q^{n+1}} \right) (x-a)^{n+1} .$$

**Theorem 4.** *(Refinement of Ostrowski inequality for q-calculus on q-lattice) Let $f : [a,b] \to \mathbb{R}$ be a q-integrable function over $[a,b]$ and $m \in \mathbb{N}_0$. Then the following inequalities hold*

$$\begin{aligned}
&\left| f(a + q^m (b-a)) - \frac{1}{b-a} \int_a^b f(t) \, d_q^a t \right| \\
&\leq (b-a) \left[ \frac{q^{2m+1}}{1+q} \left\| D_q^a f \right\|_\infty^{[a, a+q^m(b-a)]} + \left( \frac{1+q^{2m+1}}{1+q} - q^m \right) \left\| D_q^a f \right\|_\infty^{[a,b]} \right] \\
&\leq (b-a) \left( \frac{1+2q^{2m+1}}{1+q} - q^m \right) \left\| D_q^a f \right\|_\infty^{[a,b]} .
\end{aligned}$$

*Proof.* Starting from (7) we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, d_q^a \, t \right| \\
&= (b-a) \left| \int_0^{\frac{x-a}{b-a}} (qt) \, D_q^a f(tb + (1-t) a) \, d_q^0 t + \int_{\frac{x-a}{b-a}}^1 (qt-1) \, D_q^a f(tb + (1-t) a) \, d_q^0 t \right| \\
&\leq (b-a) \left| \int_0^{\frac{x-a}{b-a}} (qt) \, D_q^a f(tb + (1-t) a) \, d_q^0 t \right| \\
&+ (b-a) \left| \int_{\frac{x-a}{b-a}}^1 (qt-1) \, D_q^a f(tb + (1-t) a) \, d_q^0 t \right|
\end{aligned}$$

In order to apply triangle inequality for $q$-integrals (5) and inequality (6) we have to take $x = a + q^m (b-a)$. Thus, for $x = a + q^m (b-a)$, we further have

$$\left| \int_0^{\frac{x-a}{b-a}} (qt) \, D_q^a f(tb + (1-t) a) \, d_q^0 t \right| \leq \int_0^{\frac{x-a}{b-a}} \left| (qt) \, D_q^a f(tb + (1-t) a) \right| d_q^0 t$$

$$\leq \left\| D_q^a f \right\|_\infty^{[a,x]} \int_0^{\frac{x-a}{b-a}} qt d_q^0 t \leq \frac{q}{1+q} \left( \frac{x-a}{b-a} \right)^2 \left\| D_q^a f \right\|_\infty^{[a,x]},$$

since

$$\left\| D_q^a f \left( tb + (1-t) a \right) \right\|_\infty^{\left[0, \frac{x-a}{b-a}\right]} = \left\| D_q^a f \right\|_\infty^{[a,x]}.$$

We also note that it is valid

$$\left\| D_q^a f \left( tb + (1-t) a \right) \right\|_\infty^{[0,1]} = \left\| D_q^a f \right\|_\infty^{[a,b]}.$$

Now we obtain

$$\left| \int_{\frac{x-a}{b-a}}^{1} (qt-1) D_q^a f \left( tb + (1-t) a \right) d_q^0 t \right|$$

$$\leq \int_{\frac{x-a}{b-a}}^{1} \left| (qt-1) D_q^a f \left( tb + (1-t) a \right) \right| d_q^0 t \leq \left\| D_q^a f \right\|_\infty^{[a,b]} \int_{\frac{x-a}{b-a}}^{1} (1-qt) d_q^0 t$$

$$\leq \left\| D_q^a f \right\|_\infty^{[a,b]} \left( \left( 1 - \frac{x-a}{b-a} \right) - \frac{q}{1+q} \left( 1 - \left( \frac{x-a}{b-a} \right)^2 \right) \right).$$

If we take $x = a + q^m (b-a)$ we obtain

$$\left( 1 - \frac{x-a}{b-a} \right) - \frac{q}{1+q} \left( 1 - \left( \frac{x-a}{b-a} \right)^2 \right)$$

$$= (1 - q^m) - \frac{q}{1+q} \left( 1 - q^{2m} \right) = \frac{1+q^{2m+1}}{1+q} - q^m,$$

and the first inequality is proved. The second follows immediately after applying $\left\| D_q^a f \right\|_\infty^{[a,x]} \leq \left\| D_q^a f \right\|_\infty^{[a,b]}$. □

**Remark 4.** *Let us note, as stated at beginning of Section 4, that the function $f : [a,b] \to \mathbb{R}$ is taken to be q-differentiable on $[a,b]$. Therefore, by Remark 4 from [2], we also note that this function is continuous at $x = a$. This is the reason why we didn't write this assumption in Theorem 4. Furthermore, as for such function it is also valid*

$$\sup_{t \in [a,b]} |f(t)| = \sup_{t \in \langle a,b]} |f(t)|,$$

*we did not use the $\| \ \|_\infty^{\langle a,b]}$ norm on half-opened interval in Theorem 3 and Theorem 4, as was done in [2] in Theorem 8 and Theorem 14.*

**Remark 5.** *Second inequalities in Theorem 3 and Theorem 4 were obtained in [2] so the first ones are their refinements. Authors of [2], have also proven that they have obtained sharp inequalities. We conclude that, as their refinements, the first inequalities in Theorem 3 and Theorem 4 are also sharp.*

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