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On effective approximation to quadratic numbers

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Abstract

Let p be a prime number and $|\cdot|_p$ the p-adic absolute value on $\mathbb{Q}$ and on the p-adic field $\mathbb{Q}_p$ normalized such that $|p|_p = p^{-1}$. Let ξ be a quadratic real number and α a quadratic p-adic number. We prove that there exist positive, effectively computable, real numbers $c_1 = c_1(\xi)$, $\tau_1 = \tau_1(\xi)$, $c_2 = c_2(\alpha)$, $\tau_2 = \tau_2(\alpha)$, such that

$$|y\xi - x| \cdot |y|_p \geq c_1 |y|^{-2+\tau_1}, \text{ for } x, y \in \mathbb{Z}_{\neq 0},$$

and

$$|b\alpha - a|_p \geq c_2 |ab|^{-2+\tau_2}, \text{ for } a, b \in \mathbb{Z}_{\neq 0}.$$

Both results improve the effective lower bounds which follow from an easy Liouville-type argument.

Keywords: rational approximation, quadratic number, p-adic number, linear forms in logarithms
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1 Introduction

Let ξ be an irrational, real, algebraic number of degree d. Liouville [12] proved that there exists a positive number $c_1 = c_1(\xi)$, which can be given explicitly, such that

$$|y\xi - x| > c_1 |y|^{-d+1}, \text{ for } x, y \in \mathbb{Z}_{\neq 0}. \tag{1}$$

Since, by the theory of continued fractions, there are integers x, y with $|y|$ arbitrarily large such that $|y\xi - x| < |y|^{-1}$, Liouville's result is best possible when $d = 2$. For $d \geq 3$, after earlier works by Thue, Siegel, Gelfond, and Dyson, (1) was considerably improved by Roth [18], who showed that, for every $\varepsilon > 0$, there exists a positive number $c_2 = c_2(\xi, \varepsilon)$ such that

$$|y\xi - x| > c_2 |y|^{-1-\varepsilon}, \text{ for } x, y \in \mathbb{Z}_{\neq 0}. \tag{2}$$

However, his method of proof does not yield an explicit value for c_2. Usually, instead of saying that some constant can be given explicitly, we simply say that it is effectively computable. In the sequel, we use the notation $\gg^{\text{eff}}_{a,b,\ldots}$ and $\ll^{\text{eff}}_{a,b,\ldots}$ (resp., $\gg^{\text{ineff}}_{a,b,\ldots}$ and $\ll^{\text{ineff}}_{a,b,\ldots}$) to indicate that the positive numerical constants implied by $\gg$ and $\ll$ can be given explicitly (resp., cannot be given explicitly from the known proofs) and depend at most on the parameters $a, b, \ldots$. To give an effective version of Roth's theorem is an outstanding open problem.

In this direction, by using the theory of linear forms in logarithms developed by A. Baker in a series of papers starting with [3], Feldman [11] (see also [4]) proved that, for $d \geq 3$, there exists a (small) positive, effectively computable $\tau = \tau(\xi)$, such that

$$|y\xi - x| \gg^{\text{eff}}_{\xi} |y|^{-d+1+\tau}, \text{ for } x, y \in \mathbb{Z}_{\neq 0}. \tag{3}$$

This is stronger than (1). A natural question is then to find a way, in the case $d = 2$, to obtain stronger effective and ineffective statements, but which are valid for only a restricted class of integer pairs (x, y).

This was done by Ridout [16], whose main theorem yields the following result. For a prime number p, the field $\mathbb{Q}_p$ of p-adic numbers is equipped with the ultrametric absolute value $|\cdot|_p$ normalized such that $|p|_p = p^{-1}$.

Theorem 1 (Ridout)**.** *Let ξ be a real, irrational, algebraic number. Let $p_1, \ldots, p_t$ denote distinct prime numbers. For every $\varepsilon > 0$, we have*

$$|y\xi - x| \cdot |xy|_{p_1} \ldots |xy|_{p_t} \gg^{\text{ineff}}_{\xi, p_1, \ldots, p_t, \varepsilon} |y|^{-1-\varepsilon}, \textit{ for } x, y \in \mathbb{Z}_{\neq 0}. \tag{4}$$

Since we can assume that $|y\xi - x| < |x|/2$ in (4), the term $|y|$ can be replaced by $\max\{|x|, |y|\}$ to get a more symmetric (but equivalent) statement. Theorem 1 is clearly stronger than (2) since $|xy|_{p_1} \ldots |xy|_{p_t} \leq 1$.

Regarding approximation to an irrational, algebraic, real number ξ by rational numbers whose denominator is a power of a given integer $b \geq 2$, Theorem 1 implies that, for every $\varepsilon > 0$, we have

$$\|b^n \xi\| \gg^{\text{ineff}}_{\xi, b, \varepsilon} b^{-\varepsilon n}, \text{ for } n \geq 1.$$

Here and below, $\|\cdot\|$ denotes the distance to the nearest integer. If ξ is of degree d at least 3, then (3) gives an effective improvement over the lower bound $\|b^n\xi\| \gg_{\xi,b}^{\mathrm{eff}} b^{-(d-1)n}$, which follows from (1). It remains for us to consider the quadratic case. This was first addressed by Schinzel [19], who established that, for every integer $b \geq 2$ and every quadratic real number ξ, we have

$$\|b^n\xi\| > b^{-n}\exp\bigl(cn^{1/7}\bigr), \text{ for } n \geq 1,$$

where $c = c(\xi, b)$ is a positive, effectively computable, real number.

By following his approach and applying refined estimates of linear forms in logarithms, Bennett and Bugeaud [5] derived the stronger lower bound

$$\|b^n\xi\| \gg_{\xi,b}^{\mathrm{eff}} b^{-(1-\tau)n}, \text{ for } n \geq 1, \tag{5}$$

where $\tau = \tau(\xi, b)$ is a positive, effectively computable, real number.

Our first result is an effective version of a (much) weaker form of Theorem 1.

Theorem 2. *Let ξ be a real, irrational, algebraic number of degree d. Let $p_1, \ldots, p_t$ denote distinct prime numbers. There exists an effectively computable, positive number $\tau = \tau(\xi, p_1, \ldots, p_t)$ such that*

$$|y\xi - x| \cdot |y|_{p_1} \ldots |y|_{p_t} \gg_{\xi,p_1,\ldots,p_t}^{\mathrm{eff}} |y|^{-d+\tau}, \textit{ for } x, y \in \mathbb{Z}_{\neq 0}. \tag{6}$$

Since $|y| \cdot |y|_{p_1} \ldots |y|_{p_t} \geq 1$, Theorem 2 immediately follows from (3) when ξ has degree at least 3. Thus, it only remains for us to prove (6) for quadratic numbers ξ. This is done in Subsection 3.1.

By taking $y = b^n$ in (6) and letting $p_1, \ldots, p_t$ be the prime factors of b, we see that Theorem 2 implies (5).

We now focus our attention on the approximation to an irrational p-adic number α. It follows from the Dirichlet box principle that, for every positive integers A, B, there are integers a, b, not both 0, such that (below, we choose to simply write $\ll_\alpha$ and not $\ll_{\alpha,p}$, since p is implicitly given when α is given)

$$|b\alpha - a|_p \ll_\alpha (AB)^{-1}, \quad |a| \leq A, \ |b| \leq B.$$

Consequently, there exist infinitely many pairs of nonzero integers (a, b) such that

$$|b\alpha - a|_p \ll_\alpha (\max\{|a|, |b|\})^{-2},$$

and infinitely many pairs of nonzero integers (a, b) such that

$$|b\alpha - a|_p \ll_\alpha |ab|^{-1}.$$

In the sequel, we assume that α is algebraic of degree $d \geq 2$ and state what can be viewed as the p-adic analogue of Theorem 2.

It follows from a Liouville-type argument that

$$|b\alpha - a|_p \gg_\alpha^{\text{eff}} (\max\{|a|, |b|\})^{-d}, \text{ for } a, b \in \mathbb{Z}_{\neq 0}. \tag{7}$$

Unlike in the real case, the fact that $|b\alpha - a|_p$ is small does not imply that $|a|$ and $|b|$ are comparable. Take for instance the p-adic number

$$\alpha = p + p^3 + p^{3^2} + \ldots = \sum_{h \geq 0} p^{3^h},$$

and define

$$a_H = \sum_{h=0}^{H} p^{3^h}, \quad b_H = 1, \ H \geq 1.$$

Then, we get

$$|b_H\alpha - a_H|_p = p^{-3^{H+1}} \asymp a_H^{-3} \asymp (a_H b_H)^{-3}, \quad H \geq 1,$$

where the notation $\asymp$ means that both inequalities $\ll$ and $\gg$ hold.

Since $\max\{|a|, |b|\} \leq |ab|$ for every nonzero integers a, b, we deduce from (7) that

$$|b\alpha - a|_p \gg_\alpha^{\text{eff}} |ab|^{-d}, \text{ for } a, b \in \mathbb{Z}_{\neq 0}. \tag{8}$$

The ineffective result obtained by Ridout's theorem [17] yields considerably stronger (but ineffective) inequalities than (7) and (8).

Theorem 3 (Ridout)**.** *Let p be a prime number and α a p-adic, irrational, algebraic number. For every $\varepsilon > 0$, we have*

$$|b\alpha - a|_p \gg_{\alpha,\varepsilon}^{\text{ineff}} (\max\{|a|, |b|\})^{-2-\varepsilon}, \textit{ for } a, b \in \mathbb{Z}_{\neq 0},$$

and

$$|b\alpha - a|_p \gg_{\alpha,\varepsilon}^{\text{ineff}} |ab|^{-1-\varepsilon}, \textit{ for } a, b \in \mathbb{Z}_{\neq 0}. \tag{9}$$

As pointed out to me by Jan-Hendrik Evertse, the assertion (9) follows from [17, Eq. (2)] by taking $\zeta = 0$ and $\zeta_1 = \alpha$.

As above, the theory of linear forms in logarithms allows one to improve the Liouville bound. It yields that, if α is of degree d at least 3, then there exists a positive, effectively computable $\tau = \tau(\alpha)$ such that

$$|b\alpha - a|_p \gg_\alpha^{\text{eff}} (\max\{|a|, |b|\})^{-d+\tau} \gg_\alpha^{\text{eff}} |ab|^{-d+\tau}, \text{ for } a, b \in \mathbb{Z}_{\neq 0},$$

see [20, Section V.2]. However, (7) is best possible when α is quadratic. Our next result gives an effective improvement on (8).

Theorem 4. *Let p be a prime number and α be a quadratic p-adic number. Then, there exists a positive, effectively computable number $\tau = \tau(\alpha)$ such that*

$$|b\alpha - a|_p \gg_\alpha^{\text{eff}} |ab|^{-2+\tau}, \textit{ for } a, b \in \mathbb{Z}_{\neq 0}.$$

Theorem 4 is a special case of the following result which is stated in a slightly different way. Let p be a prime number and K an algebraic number field. Let O_K denote the ring of integers of K. Let $\mathfrak{p}$ be a prime ideal of the ring of integers in K lying above p, and denote by $e_\mathfrak{p}$ its ramification index. For a non-zero algebraic number ξ in K, let $\mathrm{v}_\mathfrak{p}(\xi)$ denote the exponent of $\mathfrak{p}$ in the decomposition of the fractional ideal ξO_K in a product of prime ideals and set

$$\mathrm{v}_p(\xi) = \frac{\mathrm{v}_\mathfrak{p}(\xi)}{e_\mathfrak{p}}.$$

This defines a valuation v_p on K which extends the p-adic valuation v_p on $\mathbb{Q}$ normalized in such a way that $\mathrm{v}_p(p) = 1$. For a nonzero element ξ in K, we set $|\xi|_p = p^{-\mathrm{v}_p(\xi)}$.

Theorem 5. *Let ξ be a quadratic complex number. Let $p_1, \ldots, p_t$ denote distinct prime numbers. There exists an effectively computable, positive number $\tau = \tau(\xi, p_1, \ldots, p_t)$ such that*

$$|y\xi - x|_{p_1} \ldots |y\xi - x|_{p_t} \gg_{\xi, p_1, \ldots, p_t}^{\text{eff}} |xy|^{-2+\tau}, \textit{ for } x, y \in \mathbb{Z}_{\neq 0}.$$

The proof of Theorem 5 is, in some respect, dual to that of Theorem 2. It also heavily depends on Baker's theory of linear forms in logarithms. Archimedean estimates are used in the proof of Theorem 5, while that of Theorem 2 depends on non-Archimedean estimates.

As expected, there is a big gap between the effective and the ineffective results in all of these Diophantine questions.

Theorems 2 and 4 were partly motivated by the works [13, 15], in which the products

$$|y| \cdot |y\xi - x| \cdot |y|_p \quad \text{and} \quad |ab| \cdot |b\alpha - a|_p$$

have been already studied, but in a different direction. Namely, de Mathan and Teulié [15], inspired by the Littlewood conjecture in Diophantine approximation, formulated the following (still open) problem, often referred to as the p-adic Littlewood conjecture. In Problems 1 and 2 below, the numbers ξ and α are not supposed to be algebraic.

Problem 1. *Let p be a prime number and ξ a real number. Is it true that*

$$\inf_{y\geq 1} y\cdot \|y\xi\|\cdot |y|_p = 0 \tag{10}$$

always holds?

Partial results towards Problem 1 have been established in [2, 9, 15]. In particular, it has been proved in [15] that a stronger form of (10) holds for real quadratic numbers ξ. According to de Mathan [14], it may be preferable to ask a slightly weaker problem than Problem 1, namely whether

$$\inf_{\substack{x\neq 0\\ y\geq 1}} y\cdot |y\xi - x|\cdot |xy|_p = 0$$

always holds. Note that in (6) with $t=1$, we can replace $|y|_{p_1}$ by $|xy|_{p_1}$.

The next (and still open) problem was first studied by de Mathan [13].

Problem 2. *Let p be a prime number and α a nonzero p-adic number. Is it true that*

$$\inf_{\substack{a\neq 0\\ b\neq 0}} |ab|\cdot |b\alpha - a|_p = 0 \tag{11}$$

always holds?

De Mathan [13] proved that a stronger form of (11) holds for quadratic p-adic numbers α. Several additional results have been subsequently obtained in [1, 14].

In [5], for every integer $b\geq 2$ and every given $\varepsilon>0$, the authors have constructed explicit examples of quadratic real numbers ξ such that

$$\|b^n\xi\| \gg b^{-\varepsilon n}, \text{ for } n\geq 1.$$

In view of this result and of Theorems 2 and 4, we propose the following open questions.

Problem 3. *Let $\varepsilon>0$ be a given real number. Let $p_1,\dots,p_t$ be distinct prime numbers. To find explicit examples of quadratic real numbers ξ such that*

$$|y\xi - x|\cdot |y|_{p_1}\dots |y|_{p_t} \gg^{\mathrm{eff}}_{\xi,p_1,\dots,p_t,\varepsilon} |y|^{-1-\varepsilon}, \textit{ for } x,y\in\mathbb{Z}_{\neq 0}.$$

Problem 4. *Let $\varepsilon > 0$ be a given real number. To find explicit examples of quadratic p-adic numbers α such that*

$$|b\alpha - a|_p \gg^{\mathrm{eff}}_{\alpha,\varepsilon} |ab|^{-1-\varepsilon}, \text{ for every } a, b \in \mathbb{Z}_{\neq 0}.$$

The sequel of the paper is organized as follows. In Section 2, we gather estimates from the theory of linear forms in logarithms. The proofs of our theorems are given in Section 3.

2 Auxiliary results

In this section, we recall, in a simplified form, estimates from the theory of linear forms in logarithms. The first one is an immediate consequence of results of Waldschmidt [21, 22]; see also [6, Theorems 2.1 and 2.2].

As usual, $h(\alpha)$ denotes the (logarithmic) Weil height of the algebraic number α.

Theorem 6. *Let $n \geq 1$ be an integer. Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers. Let $b_1, \ldots, b_n$ be integers with $b_n \neq 0$. Let $A_1, \ldots, A_n$ be real numbers with*

$$\log A_j \geq \max\{h(\alpha_j), 2\}, \quad 1 \leq j \leq n.$$

Set

$$B' = \max\Big\{3, \max_{1\leq j\leq n-1}\Big\{\frac{|b_n|}{\log A_j} + \frac{|b_j|}{\log A_n}\Big\}\Big\}.$$

Assume that

$$\Lambda := \alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1 \neq 0.$$

Then, there exists an effectively computable positive number c_1, depending only on n and on the degree over $\mathbb{Q}$ of the number field $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, such that we have

$$\log |\Lambda| \geq -c_1 \log A_1 \ldots \log A_n \log B'.$$

The second estimate was established in [8].

Theorem 7. *Let p be a prime number. Let α_1 and α_2 be multiplicatively independent algebraic numbers with $\mathrm{v}_p(\alpha_1) = \mathrm{v}_p(\alpha_2) = 0$. Let A_1 and A_2 be real numbers with*

$$\log A_j \geq \max\{h(\alpha_j), 2\}, \quad j = 1, 2.$$

Let b_1 and b_2 be positive integers and set

$$B' = \max\Big\{3, \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}\Big\}.$$

Then, there exists an effectively computable positive number c_2, depending only on the degree D over $\mathbb{Q}$ of the number field $\mathbb{Q}(\alpha_1, \alpha_2)$, such that

$$\mathrm{v}_p(\alpha_1^{b_1} - \alpha_2^{b_2}) \le c_2 p^D (\log A_1)(\log A_2)(\log B')^2. \tag{12}$$

In comparison with more classical estimates of lower bounds for linear forms in logarithms, the crucial point in Theorem 6 (and similarly in Theorem 7) is the replacement of the term $B = \max\{3, |b_1|, \ldots, |b_n|\}$ by B'. Clearly, B' is much smaller than B when $b_n = 1$ and $\log A_n$ is large, which is precisely the situation occurring in Section 3; see also [7] for more explanations and further examples of Diophantine questions where the refined estimate with B' appears to be crucial. The square in (12) is irrelevant for our purpose.

Moreover, we recall [6, Theorem 2.8] in a simplified form.

Theorem 8. *Let p be a prime number, $b \ge 2$ an integer and α a complex algebraic number of degree D. If α^b is not equal to 1, then*

$$\mathrm{v}_p(\alpha^b - 1) \ll D p^D h(\alpha)(\log b).$$

3 Proofs

Throughout this section, all the numerical constants are effective, but we do not indicate this.

3.1 Proof of Theorem 2

We adapt Schinzel's argument [19] and the proof of [5, Theorem 1.2]. It is sufficient to treat the case $\xi = \sqrt{a}$, where $a \ge 2$ is a non-square integer.

Let x, y be positive integers with $y \ge 3$ and $|x - \sqrt{a}y| \le 1$. Without any loss of generality, we assume that x and y are coprime. Write

$$y = A p_1^{\ell_1} \ldots p_t^{\ell_t},$$

where $\gcd(A, p_1 \ldots p_t) = 1$. We may assume that $A \le y^{1/2}$, since otherwise we get

$$|y| \cdot |y|_{p_1} \ldots |y|_{p_t} \gg_a |y|^{1/2}$$

and the theorem follows. Then, there exists i with $1 \le i \le t$ such that

$$\ell_i \gg_S \log y, \tag{13}$$

where S denotes the set of primes $\{p_1, \ldots, p_t\}$.

Write

$$x^2 - ay^2 = (x + \sqrt{a}y) \cdot (x - \sqrt{a}y) =: \Delta.$$

Let η denote the fundamental unit of the quadratic field generated by $\sqrt{a}$. Recall that its Galois conjugate is $\pm\eta^{-1}$. Write

$$x - \sqrt{a}y = \delta\eta^m, \quad x + \sqrt{a}y = \pm\delta^\sigma\eta^{-m},$$

where δ in $\mathbb{Q}(\sqrt{a})$ and the integer m are such that

$$|\Delta|^{1/2} \cdot \eta^{-1/2} < |\delta| \le |\Delta|^{1/2} \cdot \eta^{1/2}, \quad \delta\delta^\sigma = \Delta. \tag{14}$$

Here and below, δ^σ denotes the Galois conjugate of δ. Observe that $m \le 0$ and

$$\frac{-2\sqrt{a}y}{x + \sqrt{a}y} = \frac{x - \sqrt{a}y}{x + \sqrt{a}y} - 1 = \pm\frac{\delta}{\delta^\sigma}\eta^{2m} - 1. \tag{15}$$

If $m = 0$, then we immediately derive that $\log y \ll_{a,S} \log |2\Delta|$.

Set $M := \max\{2, |m|\}$. Since $|\delta^\sigma| \ge \eta^{-1/2}$ and

$$|\delta^\sigma\eta^{-m}| = |x + \sqrt{a}y| \ll_a y$$

we get

$$M \ll_a \log y.$$

If $|\Delta| = 1$ and $m < 0$, then $\delta = \pm 1$ and we deduce from (13) and Theorem 8 that

$$\log y \ll_S \ell_i \le \mathrm{v}_{p_i}(\eta^{2m} \pm 1) \ll_{a,S} \log M \ll_{a,S} \log\log y,$$

which gives an effective upper bound for y, depending only on a and S.

We assume now that $|\Delta| \ge 2$ and $m < 0$. Note that $\mathrm{v}_{p_i}(\delta) = 0$ since x and y are coprime. It follows from (14) that $\eta^{-1} \le |\delta/\delta^\sigma| \le \eta$. Since $\delta\Delta/\delta^\sigma$ is an algebraic integer, the height of δ^σ/δ is $\ll_a \log |\Delta|$. Furthermore, δ^σ/δ is not a unit, since $|\Delta| \ge 2$. Consequently, η and δ^σ/δ are multiplicatively independent. By Theorem 7, we get

$$\mathrm{v}_{p_i}\Big(\frac{\delta^\sigma}{\delta} \pm \eta^{2m}\Big) \le \mathrm{v}_{p_i}\Big(\Big(\frac{\delta^\sigma}{\delta}\Big)^2 - \eta^{4m}\Big) \ll_{a,p_i} (\log|\Delta|)\Big(\log\Big(2 + \frac{M}{\log|\Delta|}\Big)\Big)^2.$$

On the other hand, we derive from (13) and (15) that

$$\mathrm{v}_{p_i}\Big(\frac{\delta^\sigma}{\delta} \pm \eta^{2m}\Big) = \ell_i \gg_{a,S} \log y.$$

By combining the last two inequalities, we get

$$\log y \ll_{a,S} (\log|\Delta|)\Big(\log\Big(2+\frac{M}{\log|\Delta|}\Big)\Big)^2,$$

and, since $M \ll_a \log y$, we obtain that

$$\log y \ll_{a,S} \log|\Delta|.$$

Taking for x the nearest integer to $\sqrt{a}y$, all this proves the existence of an effectively computable positive real number κ, depending only on a and $p_1,\dots,p_t$, such that

$$\|y\sqrt{a}\| = |x-\sqrt{a}y| \geq \frac{|\Delta|}{2\sqrt{a}y+1} \gg_{a,S} y^{\kappa-1}.$$

This concludes the proof of the theorem, since $|y|\cdot|y|_{p_1}\dots|y|_{p_t} \geq 1$.

3.2 Proof of Theorems 4 and 5

We begin with the proof of Theorem 4. Let α be a quadratic p-adic number. Let ξ and ξ^σ be the complex roots of its minimal defining polynomial $P_\alpha(X)$ over the integers. In this section, the superscript $^\sigma$ denotes the Galois conjugation in the quadratic field $\mathbb{Q}(\xi)$. Let S be the set of places on $\mathbb{Q}(\xi)$ composed of the infinite places and the finite places above the prime number p. Since α is assumed to be in $\mathbb{Q}_p$, the polynomial $P_\alpha(X)$ has two roots in $\mathbb{Q}_p$. Consequently, the rank of the group O_S^* of S-units in $\mathbb{Q}(\xi)$ is equal to 3 is ξ is a real number and to 2, otherwise. We assume that ξ is real, the other case being similar.

Let a,b be nonzero integers with $|ab|\geq 2$. Assume first that $|a|\geq|b|$. If $|a|<|b|^2$, then $|ab|>|a|^{3/2}$ and, by (7), we get

$$|b\alpha-a|_p \gg_\alpha |a|^{-2} \gg_\alpha |ab|^{-4/3},$$

which is stronger than asserted. Consequently, we assume that $|a|\geq|b|^2$.

We also assume that $|b\alpha-a|_p \leq |ab|^{-3/2}$, otherwise there is nothing to prove. This implies that the norm of $b\xi-a$ is divisible by a large power of p, thus that the S-norm $N_S(b\xi-a)$ (the part of the norm composed of the primes outside S; see [10, Section 1.7] for a precise definition) of $b\xi-a$ is small. Our goal is to show that $N_S(b\xi-a)$ is not too small.

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be a fundamental system of S-units in $\mathbb{Q}(\xi)$. By [10, Proposition 4.3.12], there exist η in $\mathbb{Q}(\xi)$ and integers m_1, m_2, m_3 such that

$$b\xi - a = \eta \varepsilon_1^{m_1} \varepsilon_2^{m_2} \varepsilon_3^{m_3}, \quad \Big| h(\eta) - \frac{1}{2} \log N_S(b\xi - a) \Big| \ll_\alpha 1,$$

$$|m_1|, |m_2|, |m_3| \ll_\alpha \log |a|.$$

The last assertion follows from the proof of [10, Proposition 4.3.11] combined with [10, Proposition 4.3.9 (iii)]. Since $|a| \geq |b|^2$, we get

$$\Big| \frac{b\xi - a}{b\xi^\sigma - a} - 1 \Big| = \Big| \frac{b(\xi - \xi^\sigma)}{b\xi^\sigma - a} \Big| \ll_\alpha |a|^{-1/2}.$$

On the other hand, we have

$$\Lambda := \frac{b\xi - a}{b\xi^\sigma - a} - 1 = \frac{\eta}{\eta^\sigma} \Big(\frac{\varepsilon_1}{\varepsilon_1^\sigma} \Big)^{m_1} \Big(\frac{\varepsilon_2}{\varepsilon_2^\sigma} \Big)^{m_2} \Big(\frac{\varepsilon_3}{\varepsilon_3^\sigma} \Big)^{m_3} - 1.$$

We apply Theorem 6 to Λ to derive that

$$\log |a| \ll_\alpha (-\log |\Lambda|) \ll_\alpha \log N_S(b\xi - a) \, \log \frac{\log |a|}{\log N_S(b\xi - a)}.$$

Consequently, there exists a positive, effectively computable real number κ_1 such that

$$N_S(b\xi - a) \gg_\alpha |a|^{\kappa_1}.$$

If $|b| \geq |a|$, we proceed in a similar way. If $|b| < |a|^2$, then the desired result follows from (7). Thus, we assume that $|b| \geq |a|^2$, which implies that

$$\Big| \frac{\xi^\sigma (b\xi - a)}{\xi (b\xi^\sigma - a)} - 1 \Big| = \Big| \frac{a(\xi - \xi^\sigma)}{\xi (b\xi^\sigma - a)} \Big| \ll_\alpha |b|^{-1/2}.$$

Note that $|m_1|, |m_2|, |m_3|$ are now $\ll_\alpha \log |b|$. Arguing as above, we derive from Theorem 6 that there exists a positive, effectively computable κ_2 such that

$$N_S(b\xi - a) \gg_\alpha |b|^{\kappa_2}.$$

Consequently, we have established that there exists a positive, effectively computable κ such that

$$N_S(b\xi - a) \gg_\alpha (\max\{|a|, |b|\})^\kappa.$$

This implies that

$$|b\alpha - a|_p \gg_\alpha (\max\{|a|, |b|\})^{-2+\kappa} \gg_\alpha |ab|^{-2+\kappa},$$

and the proof of Theorem 4 is complete.

For the proof of Theorem 5, we proceed in a very similar way. We take for S the set of places on $\mathbb{Q}(\xi)$ composed of the infinite places and the finite places above the prime numbers $p_1, \dots, p_t$. We conclude that $N_S(b\xi - a)$ exceeds some positive constant times a small power of $\max\{|a|, |b|\}$. This implies Theorem 5.

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