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Positive operator frame for Hilbert spaces

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Abstract

Motived by the characterization of the positive elements in a C^*–algebra and the decomposition of an operator into a sum of orthogonal projections, we introduce the notions of positive operator and K-operator frame for $B(\mathcal{H})$. Also, we give some properties.

Keywords: Frame, Positive operator, Hilbert space
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1 Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaefer [3] in 1952 to study some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [5] for signal processing. In fact, in 1946 Gabor, showed that any function $f \in L^2(\mathbb{R})$ could be reconstructed via a Gabor system $\{g(x-ka)e^{2\pi imbx} : k, m \in \mathbb{Z}\}$ where g is a continuous compact support function. These ideas did not generate much interest outside of nonharmonic Fourier series and signal processing until the landmark paper of Daubechies, Grossmann, and Meyer [4] in 1986, where they developed the class of tight frames for signal reconstruction. They showed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$, which are very similar to the expansions using orthonormal bases. After this innovative work, the theory of frames began to be widely studied. While orthonormal bases have been widely used for many applications [7], it is the redundancy that makes frames useful in applications.

The last decades have seen tremendous activity in the development of frame theory, and many generalizations of frames have come into existence in Hilbert Spaces and Hilbert C^*-modules [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

In this paper, we introduce the notions of positive operator and K-operator frame for $B(\mathcal{H})$, and we establish her relation with g-frame, K-g-frame, operator frame, and K-operator frame.

2 Preliminaries

Throughout the paper, $\mathcal{H}$ denotes a separable Hilbert space and $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is called positive if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and the set of all positive operators is denoted by $B(\mathcal{H})^+$. Let I be a finite or countable index subset of $\mathbb{N}$.

This section recalls the definitions of g-frame, K-g-frame, operator frame, and K-operator frame. For more on frames in Hilbert spaces, see [2].

Definition 1. *[20] We call a sequence $\{\Lambda_i \in B(\mathcal{H}, V_i) : i \in I\}$ a g-frame for the Hilbert space $\mathcal{H}$ with respect to $\{V_i : i \in I\}$ if there exist positive constants $A, B > 0$ such that for all $x \in \mathcal{H}$,*

$$A\|x\|^2 \leq \sum_{i\in I} \|\Lambda_i x\|^2 \leq B\|x\|^2.$$

The numbers A and B are called g-frame bounds. If $A = B = \lambda$, the g-frame is λ-tight. If $A = B = 1$, it is called a Parseval g-frame. If only the second inequality holds, we call it a g-Bessel sequence.

Definition 2. *[1]Let $K \in B(\mathcal{H})$. A sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\}$ is called a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$, if there exist constants $A, B > 0$ such that*

$$A\|K^*x\|^2 \leq \sum_{i\in I} \|\Lambda_i x\|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

The constants A and B are called lower and upper bounds for the K-g-frame, respectively. A K-g-frame $\{\Lambda_i\}_{i\in I}$ is said to be tight if there exists a constant $A > 0$ such that

$$A\|K^*x\|^2 = \sum_{i\in I} \|\Lambda_i x\|^2 \quad \forall x \in \mathcal{H}. \tag{1}$$

It is called Parseval K-g-frame if $A = 1$ in (1).

Definition 3. *[6] A family of bounded linear operators $\{T_i\}_{i\in I}$ on a Hilbert space $\mathcal{H}$ is said to be an operator frame for $B(\mathcal{H})$, if there exist positive constants $A, B > 0$ such that*

$$A\|x\|^2 \leq \sum_{i\in I} \|T_i x\|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H}. \tag{2}$$

where A and B are called lower and upper bounds for the operator frame, respectively. An operator frame $\{T_i\}_{i\in I}$ is said to be tight if $A = B$. It is called Parseval operator frame if $A = B = 1$. If only upper inequality of (2) *hold, then $\{T_i\}_{i\in I}$ is called an operator Bessel sequence for $B(\mathcal{H})$.*

Definition 4. *[19] Let $K \in B(\mathcal{H})$. A family of bounded linear operators $\{T_i\}_{i\in I}$ on a Hilbert space $\mathcal{H}$ is said to be a K-operator frame for $B(\mathcal{H})$, if there exist positive constants $A, B > 0$ such that*

$$A\|K^*x\|^2 \leq \sum_{i\in I} \|T_i x\|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

where A and B are called lower and upper bounds for the K-operator frame, respectively. A K-operator frame $\{T_i\}_{i\in I}$ is said to be tight if there exists a constant$A > 0$ such that

$$A\|K^*x\|^2 = \sum_{i\in I} \|T_i x\|^2 \quad \forall x \in \mathcal{H}. \tag{3}$$

It is called Parseval K-operator frame if $A = 1$ in (3).

3 Positive Operator Frame

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert Space and $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Then $\forall x \in \mathcal{H}$, $x = \sum_{i\in\mathbb{N}}\langle x, e_i\rangle e_i = \sum_{i\in\mathbb{N}} e_i \otimes e_i(x)$ so $id_{\mathcal{H}} = \sum_{i\in\mathbb{N}} e_i \otimes e_i$. Motivated by this, we give the following definition.

Definition 5. *A family of positive operators $\{T_i\}_{i\in I}$ on a Hilbert space $\mathcal{H}$ is said to be a positive operator frame for $B(\mathcal{H})$, if there exist positive constants $A, B > 0$ such that*

$$A\|x\|^2 \leq \sum_{i\in I}\langle T_i x, x\rangle \leq B\|x\|^2 \quad \forall x \in \mathcal{H}. \tag{4}$$

The numbers A and B are called lower and upper bounds of the positive operator frame, respectively. If $A = B$, the positive operator frame is tight. If $A = B = 1$, it is called a normalized tight positive operator frame or a Parseval positive operator frame. If only upper inequality of (4) *hold, then $\{T_i\}_{i\in I}$ is called a Bessel positive operator for $B(\mathcal{H})$.*

Example 1. *Let $\mathcal{H}$ be a Hilbert Space and $(f_i)_{i\in I}$ be a frame for $\mathcal{H}$. Then there exist positive constants $A, B > 0$ such that*

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, f_i\rangle|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H}. \tag{5}$$

Define $\{T_i\}_{i\in I} \subset B(\mathcal{H})^+$ by $T_i = f_i \otimes f_i$ for all $i \in I$. Then

$$\begin{aligned}\sum_{i\in I}\langle T_i x, x\rangle &= \sum_{i\in I}\langle (f_i \otimes f_i)x, x\rangle \\ &= \sum_{i\in I}\langle \langle x, f_i\rangle f_i, x\rangle = \sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle = \sum_{i\in I} |\langle x, f_i\rangle|^2\end{aligned}$$

for all $x \in \mathcal{H}$. Then by (5), we have

$$A\|x\|^2 \leq \sum_{i\in I}\langle T_i x, x\rangle \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

Thus $\{T_i\}_{i\in I}$ is a positive operator frame for $B(\mathcal{H})$.

Theorem 1. *Every positive operator frame corresponds to an operator frame and vice versa.*

Proof. Let $\{T_i\}_{i\in I}$ be an operator frame for $B(\mathcal{H})$, then there exist positive constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{i\in I} \|T_i x\|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

Hence

$$A\|x\|^2 \leq \sum_{i\in I}\langle T_i^* T_i x, x\rangle \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

Thus $\{T_i^* T_i\}_{i\in I}$ is a positive operator frame for $B(\mathcal{H})$.

For the converse, let $\{\tilde{T}_i\}_{i\in I}$ be a positive operator frame for $B(\mathcal{H})$. Since $\tilde{T}_i \in B(\mathcal{H})^+$, there exist $T_i \in B(\mathcal{H})$ such that $\tilde{T}_i = T_i^* T_i$. $\{\tilde{T}_i\}_{i\in I}$ is a positive operator frame for $B(\mathcal{H})$, hence there exist positive constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{i\in I}\langle \tilde{T}_i x, x\rangle \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

It follows that

$$A\|x\|^2 \leq \sum_{i\in I}\langle T_i^* T_i x, x\rangle \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

Thus

$$A\|x\|^2 \leq \sum_{i\in I} \|T_i x\|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

Hence $\{T_i\}_{i\in I}$ is an operator frame for $B(\mathcal{H})$. $\square$

Corollary 1. *Every positive tight operator frame is corresponds to a tight operator frame and vice versa.*

Corollary 2. *Every Bessel positive operator corresponds to an operator Bessel sequence and vice versa.*

Theorem 2. *Every g-frame $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$, corresponds to a positive operator frame for $B(\mathcal{H})$. The convers is valid.*

Proof. Result of the characterization for the positive elements in a C^*-algebra. $\square$

Definition 6. *Let $\{T_i\}_{i\in I}$ be a positive operator frame for $B(\mathcal{H})$. We define the frame operator $S : \mathcal{H} \to \mathcal{H}$ by*

$$Sx = \sum_{i\in I} T_i x,$$

for all $x \in \mathcal{H}$.

Theorem 3. *Assume that S is the frame operator of a positive operator frame $\{T_i\}_{i\in I}$ for $B(\mathcal{H})$ with bounds A and B, then S is positive, self-adjoint and invertible. Moreover, we have $AI \leq S \leq BI$ and the reconstruction formula*

$$x = \sum_{i\in I} S^{-1} T_i x = \sum_{i\in I} T_i S^{-1} x \quad \forall x \in \mathcal{H}.$$

Proof. It is clear that S is positive and self-adjoint. For any $x \in \mathcal{H}$, since $\{T_i\}_{i\in I}$ is a positive operator frame with bounds A, B, we have

$$\langle Ax, x\rangle = A\|x\|^2 \leq \sum_{i\in I} \langle T_i x, x\rangle = \langle Sx, x\rangle \leq B\|x\|^2 = \langle Bx, x\rangle.$$

This shows that

$$AI \leq S \leq BI,$$

which implies that S is invertible. Further, for any $x \in \mathcal{H}$, we have

$$x = S^{-1} Sx = S^{-1} \sum_{i\in I} T_i x = \sum_{i\in I} S_T^{-1} T_i x,$$

and

$$x = SS^{-1}x = \sum_{i\in I} T_i S^{-1}x.$$

□

Theorem 4. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. Let $\{\Lambda_i\}_{i\in I}$, $\{\Gamma_j\}_{j\in J}$ be two positive operator frames for $B(\mathcal{H})$ and $B(\mathcal{K})$ with frame operator S_Λ and S_Γ with bounds (A, B) and (C, D) respectively. Then $\{\Lambda_i \otimes \Gamma_j\}_{i\in I, j\in J}$ is a positive operator frame for $B(\mathcal{H}\otimes\mathcal{K})$ with frame operator $S_\Lambda \otimes S_\Gamma$ and lower and upper bounds AC and BD, respectively.*

Proof. By the definition of the positive operator frame $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_j\}_{j\in J}$ we have

$$A\|x\|^2 \leq \sum_{i\in I}\langle \Lambda_i x, x\rangle \leq B\|x\|^2 \quad \forall x \in \mathcal{H},$$

$$C\|y\|^2 \leq \sum_{j\in J}\langle \Gamma_j y, y\rangle \leq D\|y\|^2 \quad \forall y \in \mathcal{K}.$$

Therefore,

$$AC\|x\|^2\|y\|^2 \leq \sum_{i\in I}\langle \Lambda_i x, x\rangle \otimes \sum_{j\in J}\langle \Gamma_j y, y\rangle \leq BD\|x\|^2\|y\|^2$$

$\forall x \in \mathcal{H}$, $\forall y \in \mathcal{K}$. Then

$$AC\|x\otimes y\|^2 \leq \sum_{\substack{i\in I\\ j\in J}}\langle \Lambda_i x, x\rangle \otimes \langle \Gamma_j y, y\rangle \leq BD\|x\otimes y\|^2$$

$\forall x \in \mathcal{H}$, $\forall y \in \mathcal{K}$. Consequently we have

$$AC\|x\otimes y\|^2 \leq \sum_{\substack{i\in I\\ j\in J}}\langle \Lambda_i x \otimes \Gamma_j y, x\otimes y\rangle \leq BD\|x\otimes y\|^2$$

$\forall x \in \mathcal{H}$, $\forall y \in \mathcal{K}$. Then $\forall x\otimes y \in \mathcal{H}\otimes\mathcal{K}$ we have

$$AC\|x\otimes y\|^2 \leq \sum_{\substack{i\in I\\ j\in J}}\langle (\Lambda_i \otimes \Gamma_j)(x\otimes y), x\otimes y\rangle \leq BD\|x\otimes y\|^2.$$

The last inequality is satisfied for every finite sum of elements in $\mathcal{H}\otimes_{alg}\mathcal{K}$ and then it is satisfied for all $z \in \mathcal{H}\otimes\mathcal{K}$. It shows that $\{\Lambda_i \otimes \Gamma_j\}_{i\in I, j\in J}$ is a positive operator frame for $B(\mathcal{H}\otimes\mathcal{K})$ with lower and upper bounds AC and BD, respectively.

By the definition of frame operator S_Λ and S_Γ we have:

$$S_\Lambda x = \sum_{i\in I} \Lambda_i x \ \ \forall x \in \mathcal{H}, \quad S_\Gamma y = \sum_{j\in J} \Gamma_j y \ \ \forall y \in \mathcal{K}.$$

Therefore,

$$\begin{aligned}(S_\Lambda \otimes S_\Gamma)(x\otimes y) = S_\Lambda x \otimes S_\Gamma y &= \sum_{i\in I} \Lambda_i x \otimes \sum_{j\in J} \Gamma_j y \\ &= \sum_{\substack{i\in I\\ j\in J}} \Lambda_i x \otimes \Gamma_j y = \sum_{\substack{i\in I\\ j\in J}} (\Lambda_i \otimes \Gamma_j)(x\otimes y).\end{aligned}$$

Now by the uniqueness of frame operator, the last expression is equal to $S_{\Lambda\otimes\Gamma}(x\otimes y)$. Consequently we have $(S_\Lambda \otimes S_\Gamma)(x\otimes y) = S_{\Lambda\otimes\Gamma}(x\otimes y)$. The last equality is satisfied for every finite sum of elements in $\mathcal{H}\otimes_{alg}\mathcal{K}$ and then it is satisfied for all $z \in \mathcal{H}\otimes\mathcal{K}$. It shows that $(S_\Lambda\otimes S_\Gamma)(z) = S_{\Lambda\otimes\Gamma}(z)$. So $S_{\Lambda\otimes\Gamma} = S_\Lambda \otimes S_\Gamma$. $\square$

4 Positive K-operator frame

Definition 7. *Let $K \in B(\mathcal{H})^+$. A family of positive operators $\{T_i\}_{i\in I}$ on a Hilbert space $\mathcal{H}$ is said to be a K-positive operator frame for $B(\mathcal{H})$, if there exist positive constants $A, B > 0$ such that*

$$A\langle Kx, x\rangle \leq \sum_{i\in I} \langle T_i x, x\rangle \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

The numbers A and B are called lower and upper bounds of the K-positive operator frame, respectively. If $A = B$, the positive K-operator frame is tight. If $A = B = 1$, it is called a normalized tight K-positive operator frame or a Parseval K-positive operator frame.

Example 2. *Let $\mathcal{H}$ be a Hilbert Space and $(f_i)_{i\in I}$ be a K-frame for $\mathcal{H}$. Then there exist positive constants $A, B > 0$ such that*

$$A\|K^* x\|^2 \leq \sum_{i\in I} |\langle x, f_i\rangle|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H}. \tag{6}$$

Define $\{T_i\}_{i\in I} \subset B(\mathcal{H})^+$ by $T_i = f_i \otimes f_i$ for all $i \in I$. Then

$$\begin{aligned}\sum_{i\in I}\langle T_i x, x\rangle &= \sum_{i\in I}\langle (f_i\otimes f_i)x, x\rangle \\ &= \sum_{i\in I}\langle\langle x, f_i\rangle f_i, x\rangle = \sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle = \sum_{i\in I}|\langle x, f_i\rangle|^2,\end{aligned}$$

for all $x \in \mathcal{H}$. Then by (6) we have

$$A\langle KK^*x, x\rangle = A\|K^*x\|^2 \leq \sum_{i\in I}\langle T_i x, x\rangle \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

Thus $\{T_i\}_{i\in I}$ is a KK^*-positive operator frame for $B(\mathcal{H})$.

Theorem 5. *Every K-positive operator frame corresponds to a K-operator frame and vice versa.*

Proof. Let $\{T_i\}_{i\in I}$ be a K-operator frame for $B(\mathcal{H})$, then there exist positive constants $A, B > 0$ such that

$$A\|K^*x\|^2 \leq \sum_{i\in I}\|T_i x\|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

Hence

$$A\langle K^*Kx, x\rangle \leq \sum_{i\in I}\langle T_i^* T_i x, x\rangle \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

Thus $\{T_i^* T_i\}_{i\in I}$ is a K^*K-positive operator frame for $B(\mathcal{H})$.

For the convese, let $\{\tilde{T}_i\}_{i\in I}$ be a $\tilde{K}$-positive operator frame for $B(\mathcal{H})$. $\tilde{T}_i \in B(\mathcal{H})^+$ then there exist $T_i \in B(\mathcal{H})$ such that $\tilde{T}_i = T_i^* T_i$. $\{\tilde{T}_i\}_{i\in I}$ is a $\tilde{K}$-positive operator frame for $B(\mathcal{H})$, then there exist positive constants $A, B > 0$ such that

$$A\langle \tilde{K}x, x\rangle \leq \sum_{i\in I}\langle \tilde{T}_i x, x\rangle \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

Hence,

$$A\langle KK^*x, x\rangle \leq \sum_{i\in I}\langle T_i^* T_i x, x\rangle \leq B\|x\|^2 \quad \forall x \in \mathcal{H},$$

thus

$$A\|K^*x\|^2 \leq \sum_{i\in I}\|T_i x\|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

Therefore, $\{T_i\}_{i\in I}$ is a K-operator frame for $B(\mathcal{H})$. □

Corollary 3. *Every positive tight K-operator frame corresponds to a tight K-operator frame and vice versa.*

Theorem 6. *Every K-g-frame $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$, corresponds to a K-positive operator frame for $B(\mathcal{H})$. The converse is also valid.*

Proof. Result of the characterization for the positive elements in a C^*-algebra. □

Definition 8. *Let $K \in B(\mathcal{H})^+$ and $\{T_i\}_{i\in I}$ be a K-positive operator frame for $B(\mathcal{H})$. We define the frame operator $S : \mathcal{H} \to \mathcal{H}$ by*

$$Sx = \sum_{i\in I} T_i x,$$

for all $x \in \mathcal{H}$.

Theorem 7. *Assume that S is the frame operator of a K-positive operator frame $\{T_i\}_{i\in I}$ for $B(\mathcal{H})$ with bounds A and B, then S is positive and self-adjoint. Moreover, we have $AK \leq S \leq BI$*

Proof. It is clear that S is positive and self-adjoint. For any $x \in \mathcal{H}$, since $\{T_i\}_{i\in I}$ is a K-positive operator frame with bounds A, B, we have

$$\langle AKx, x\rangle = A\langle Kx, x\rangle \leq \sum_{i\in I}\langle T_i x, x\rangle = \langle Sx, x\rangle \leq B\|x\|^2 = \langle Bx, x\rangle.$$

This shows that

$$AK \leq S \leq BI,$$

□

Theorem 8. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. Let $\{\Lambda_i\}_{i\in I}$, $\{\Gamma_j\}_{j\in J}$ be K-positive operator frame for $B(\mathcal{H})$ and L-positive operator frame for $B(\mathcal{K})$ with frame operator S_Λ and S_Γ with bounds (A, B) and (C, D) respectively. Then $\{\Lambda_i \otimes \Gamma_j\}_{i\in I, j\in J}$ is a $K \otimes L$-positive operator frame for $B(\mathcal{H} \otimes \mathcal{K})$ with frame operator $S_\Lambda \otimes S_\Gamma$ and lower and upper bounds AC and BD, respectively.*

Proof. By the definition of $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_j\}_{j\in J}$ we have:

$$A\langle Kx, x\rangle \leq \sum_{i\in I}\langle \Lambda_i x, x\rangle \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

$$C\langle Ly, y\rangle \leq \sum_{j\in J}\langle \Gamma_j y, y\rangle \leq D\|y\|^2 \quad \forall y \in \mathcal{K}.$$

Therefore

$$\begin{aligned} AC\langle Kx, x\rangle\langle Ly, y\rangle &\leq \sum_{i\in I}\langle \Lambda_i x, x\rangle \otimes \sum_{j\in J}\langle \Gamma_j y, y\rangle \\ &\leq BD\|x\|^2\|y\|^2 \quad \forall x \in \mathcal{H}, \forall y \in \mathcal{K}. \end{aligned}$$

Then

$$\begin{aligned} AC\langle Kx \otimes Ly, x \otimes y\rangle &\leq \sum_{i\in I, j\in J}\langle \Lambda_i x, x\rangle \otimes \langle \Gamma_j y, y\rangle \\ &\leq BD\|x \otimes y\|^2 \quad \forall x \in \mathcal{H}, \forall y \in \mathcal{K}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} AC\langle (K\otimes L)(x\otimes y), x\otimes y\rangle &\leq \sum_{i\in I, j\in J} \langle \Lambda_i x\otimes \Gamma_j y, x\otimes y\rangle \\ &\leq BD\|x\otimes y\|^2 \quad \forall x\in\mathcal{H}, \forall y\in\mathcal{K}. \end{aligned}$$

Then, for all $x\otimes y\in\mathcal{H}\otimes\mathcal{K}$ we have

$$\begin{aligned} AC\langle (K\otimes L)(x\otimes y), x\otimes y\rangle &\leq \sum_{i\in I, j\in J} \langle (\Lambda_i\otimes \Gamma_j)(x\otimes y), x\otimes y\rangle \\ &\leq BD\|x\otimes y\|^2. \end{aligned}$$

The last inequality is satisfied for every finite sum of elements in $\mathcal{H}\otimes_{alg}\mathcal{K}$ and then it's satisfied for all $z\in\mathcal{H}\otimes\mathcal{K}$. It shows that $\{\Lambda_i\otimes\Gamma_j\}_{i\in I, j\in J}$ is a $K\otimes L$-positive operator frame for $B(\mathcal{H}\otimes\mathcal{K})$ with lower and upper bounds AC and BD, respectively.

By the definition of frame operator S_Λ and S_Γ we have

$$S_\Lambda x = \sum_{i\in I}\Lambda_i x \ \ \forall x\in\mathcal{H}, \quad S_\Gamma y = \sum_{j\in J}\Gamma_j y \ \ \forall y\in\mathcal{K}.$$

Therefore,

$$\begin{aligned} (S_\Lambda\otimes S_\Gamma)(x\otimes y) = S_\Lambda x\otimes S_\Gamma y &= \sum_{i\in I}\Lambda_i x\otimes\sum_{j\in J}\Gamma_j y \\ &= \sum_{\substack{i\in I\\ j\in J}}\Lambda_i x\otimes\Gamma_j y = \sum_{\substack{i\in I\\ j\in J}}(\Lambda_i\otimes\Gamma_j)(x\otimes y). \end{aligned}$$

Now, by uniqueness of frame operator, the last expression is equal to $S_{\Lambda\otimes\Gamma}(x\otimes y)$. Consequently, we have $(S_\Lambda\otimes S_\Gamma)(x\otimes y) = S_{\Lambda\otimes\Gamma}(x\otimes y)$. The last equality is satisfied for every finite sum of elements in $\mathcal{H}\otimes_{alg}\mathcal{K}$ and then it is satisfied for all $z\in\mathcal{H}\otimes\mathcal{K}$. It shows that $(S_\Lambda\otimes S_\Gamma)(z) = S_{\Lambda\otimes\Gamma}(z)$. So $S_{\Lambda\otimes\Gamma} = S_\Lambda\otimes S_\Gamma$. □

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