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A note on a category composition

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Abstract

The special properties of an abstract category morphism (for instance, being an identity, an isomorphism, an epimorphism., a monomorphism . . .) fully depend on the category composition. Consequently, an isomorphic category to a concrete category may be not concrete, i.e., the concreteness is not a category invariant. Further, every small category is isomorphic to a small category whose objects are sets and whose morphisms are functions between those sets.

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1 Introduction

One often thinks that in a category whose objects are somehow enriched sets and the morphisms are the structure preserving set functions, a special property of a morphism (being an identity, isomorphism, epi, mono . . .) of the category is its “internal” one. However, it is an illusion, since we show that the category composition is “in charge”, i.e., it brings a special property to a morphism. In a *concrete* category only, being an identity and being an isomorphism are the internal properties.

We follow the category theory language of [2]. Hereby, we reduce the notation of a category $\mathcal{C} = (\mathcal{O}, \mathcal{M}, dom, cod, \circ)$ to $(\mathcal{O}, \mathcal{M}, \circ)$ by assuming that the morphism sets (members of the class $\mathcal{M}$) are *pairwise disjoint*. (Equivalently, a category morphism is assumed to be an ordered triple: (*dom-object, arrow, cod-object*)) Further, without loss of generality, we consider a concrete category to be a category admitting the forgetful functor whose objects are *sets* endowed with a structure and whose morphisms are *functions*, between those sets, "preserving" the structure. Of course, the cases without any structure are included, i.e., all subcategories of the category *Set* of all sets and all (set) functions.

2 Facta and proofs

Theorem 1. *Let $\mathcal{C}$ be a category having the class $Ob\mathcal{C} \equiv \mathcal{O}$ of objects, the class $Mor\mathcal{C} \equiv \mathcal{M}$ of all the morphism sets $\mathcal{C}(X,Y)$, $(X,Y) \in \mathcal{O} \times \mathcal{O}$, and the composition*

$$\circ : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z), \quad (f,g) \mapsto \circ(f,g) \equiv g \circ f,$$

$X,Y,Z \in \mathcal{O}$.Further, let $\mathcal{O}'$ be a class and let, for every ordered pair $(X',Y') \in \mathcal{O}' \times \mathcal{O}'$, a set $M(X',Y')$ be given such that these sets are pairwise disjoint. Assume that there exist a bijection

$$\Phi : \mathcal{O} \to \mathcal{O}', \quad X \mapsto \Phi(X) \equiv X',$$

and, for every ordered pair $(X,Y) \in \mathcal{O} \times \mathcal{O}$, a bijection

$$\Phi_Y^X : \mathcal{C}(X,Y) \to M(\Phi(X),\Phi(Y)) \equiv M(X',Y'), \quad f \mapsto \Phi_Y^X(f) \equiv f',$$

(i) Then the rule

$$(\forall X,Y,Z \in \mathcal{O})(\forall f \in \mathcal{C}(X,Y)(\forall g \in \mathcal{C}(Y,Z)\, \Phi_Z^Y(g) * \Phi_Y^X(f) = \Phi_Z^X(g \circ f), \tag{1}$$

defines a class of set functions

$$* : M(X',Y') \times M(Y',Z') \to M(X',Z'),$$
$$(f',g') \mapsto *(f',g') \equiv g' * f', \quad X',Y',Z' \in \mathcal{O}'$$

such that the class $\mathcal{O}'$ (as objects), the class $\mathcal{M}'$ of all sets $M(X',Y')$ (as morphisms) and "$*$" (as the composition) make a category $\mathcal{C}'$. Moreover, the bijection Φ and all the bijections Φ_Y^X induce a functor

$$F : \mathcal{C} \to \mathcal{C}', \quad X \mapsto F(X) = \Phi(X) \equiv X', \quad f \mapsto F(f) = \Phi_Y^X(f) \equiv f',$$

that is an isomorphism of the categories.

(ii) If $\sim$ *is a natural equivalence relation on* $Mor\mathcal{C}$*, then the rule*

$$(f_1' = \Phi_Y^X(f_1) \sim' \Phi_Y^X(f_2) = f_2') \Leftrightarrow (f_1 \sim f_2) \tag{2}$$

defines a natural equivalence relation on $Mor\mathcal{C}'$*, and the induced functor*

$$\tilde{F} : \mathcal{C}/\sim \to \mathcal{C}'/\sim', \quad X \mapsto \tilde{F}(X) = \Phi(X) \equiv X',$$

$$[f] \mapsto \tilde{F}([f]) = [F(f)]' = [\Phi_Y^X(f)]' \equiv [f']',$$

is an isomorphism of the quotient categories.

Proof. (i). Given $X', Y', Z' \in \mathcal{O}'$, the function $*$ is well defined because the function Φ and all the functions Φ_Y^X, $X, Y \in \mathcal{O}$, are bijective, and thus, with each element (f', g'), $(f' = \Phi_Y^X(f), g' = \Phi_Z^Y(g))$ of the domain, it is associated the unique element $g' * f' \equiv *(f', g')$ $(= \Phi_Z^X(g \circ f))$ of the codomain. Now one straightforwardly verifies the needed category and functor conditions. For instance, let $u' \in M(X', X')$ such that $X' = \Phi(X)$ and $u' = \Phi_X^X(1_X)$, and let $f' \in M(X', Y')$, $Y' \in \mathcal{O}'$. By the assumptions, then there exists a unique $f \in C(X, Y)$ such that

$$f' = \Phi_Y^X(f) \in \Phi_Y^X(\Phi(X), \Phi(Y)) = M(X', Y').$$

Then

$$f' * u' = \Phi_Y^X(f) * \Phi_X^X(1_X) = \Phi_Y^X(f \circ 1_X) = \Phi_Y^X(f) = f'.$$

Similarly, for every $g' \in M(Z', X')$, $Z' \in \mathcal{O}'$, it holds $u' * g' = g'$. Therefore, u' is the unique identity morphism $1_{X'}$ with respect to $*$.

(ii). Again by the mentioned bijectivity, the relation $\sim'$ is well defined on each set

$$M(X', Y') = \mathcal{C}'(\Phi(X), \Phi(Y)), \quad X', Y' \in \mathcal{O}' = Ob\mathcal{C}'.$$

Then the accordance to the composition $*$ follows by the same property of $\sim$ with respect to $\circ$. The rest follows straightforwardly. □

Example 1. *Let* $\mathcal{C} \subseteq Set$ *be the full subcategory determined by the objects (sets)* $A = \{1\}$ *and* $B = \{2, 3\}$*. Then, clearly,*

$$\mathcal{C}(A, A) = \{1_A \equiv u\}, \quad \mathcal{C}(A, B) = \{c_2 \equiv f_1, c_3 \equiv f_2\},$$

$$\mathcal{C}(B, A) = \{c_1 \equiv g\}, \quad \mathcal{C}(B, B) = \{1_B \equiv v_1, p \equiv v_2, c_2 \equiv v_3, c_3 \equiv v_4\},$$

where c's are the appropriate constant functions while p is the permutation. Let $\mathcal{O}' = \{A', B'\}$*, where* A' *and* B' *are sets containing at least one and two elements respectively. Further, let*

$$M(A', A') = \{u'\}, \quad M(A', B') = \{f_1', f_2'\}$$
$$M(B', A') = \{g'\}, \quad M(B', B') = \{v_1', v_2', v_3', v_4'\},$$

where the elements of all those sets are arbitrarily chosen corresponding functions. (For instance, if $card(A') \geq 2$*, then one may choose* u' *to be a non-identity function, while if* $card(B') \geq 4$*, then one may choose all v's*$_i$ *to be the constant functions.) Now, according to Theorem 1,* $\mathcal{C}' \equiv (Ob\mathcal{C}', Mor\mathcal{C}', *)$*, where* $Ob\mathcal{C}' = \mathcal{O}'$*,*

$$Mor\mathcal{C}' = \{M(X', Y') \mid X', Y' \in \{A', B'\}\}$$

and

$$*: M(X', Y') \times M(Y', Z') \to M(X', Z'),$$
$$*(s', t') \equiv s' * t' = s \circ t, \quad X', Y', Z' \in \{A', B'\},$$

is a category and

$$F : \mathcal{C} \to \mathcal{C}', \quad X \mapsto F(X) = X', \quad s \mapsto F(s) = s',$$

is a category isomorphism. So, we see that any function u' *on* A' *(*v' *on* B'*) can become the identity morphism* $1_{A'}$ *(*$1_{B'}$*) in* $\mathcal{C}'$*. Also, any function on* B' *can become an isomorphism in* $\mathcal{C}'$*. Notice that the category* $\mathcal{C}$ *is concrete, while to it isomorphic category* $\mathcal{C}'$ *is not concrete, because it does not admit the forgetful functor. Finally, notice that by choosing the other bijection of* $\mathcal{C}(A, B)$ *onto* $M(A', B')$ *and any bijection of* $\mathcal{C}(B, B)$ *onto* $M(B', B')$*, a new category isomorphism appears.*

An immediate consequence of Theorem 1 and the axiom of choice reads as follows.

Corollary 1. *Let* $\mathcal{C} = (Ob\mathcal{C}, Mor\mathcal{C}, \circ)$ *be a category, and let, for each* $X \in Ob\mathcal{C}$*, an* $u_X \in \mathcal{C}(X, X)$ *be chosen. Then there exists a composition* $*$ *on* $Mor\mathcal{C}$ *such that*

$$(Ob\mathcal{C}, Mor\mathcal{C}, *) \equiv \mathcal{C}'$$

is a category isomorphic to $\mathcal{C}$ *and the class*

$$\{1_X \mid X \in Ob\mathcal{C}\} \subseteq Mor\mathcal{C}'$$

of all identity morphism of $\mathcal{C}'$ *is the class*

$$\{u_X \mid X \in Ob\mathcal{C}\} \subseteq Mor\mathcal{C}.$$

Further, let for each ordered pair $(X, Y) \in Ob\mathcal{C} \times Ob\mathcal{C}$, *such that* X *is isomorphic to* Y *in* $\mathcal{C}$, *a morphism* $f_Y^X \in \mathcal{C}(X, Y)$ *be given. Then there exists a composition* $\bullet$ *on* $Mor\mathcal{C}$ *such that* $(Ob\mathcal{C}, Mor\mathcal{C}, \bullet) \equiv \mathcal{C}''$ *is a category isomorphic to* $\mathcal{C}$ *and to* $\mathcal{C}'$, *and each chosen*

$$f_Y^X \in \mathcal{C}''(X, Y) = \mathcal{C}(X, Y)$$

is an isomorphism of $\mathcal{C}''$.

Though Theorem 1 might seem to be artificial, its importance and usefulness confirms the following theorem which shows that every small category is isomorphic to a small category whose objects and morphisms are sets and set functions respectively. (The axiom of choice is assumed!) However, in general, it is not a concrete one.

Theorem 2. *Let* $\mathcal{C} = (Ob\mathcal{C}, Mor\mathcal{C}, \circ)$ *be a small category. Then there exist a proper subclass* $\mathcal{O}' \subsetneqq Ob(Set)$, *that is a set of sets, a proper subclass* $\mathcal{M}' \subsetneqq Mor(Set)$, *that is a set of sets of functions, and a composition* $*$ *on* $\mathcal{M}'$ *such that* $\mathcal{C}' \equiv (\mathcal{O}', \mathcal{M}', *)$ *is a category isomorphic to* $\mathcal{C}$. *In addition, one can achieve that* $\mathcal{M}'$ *does not contain any bijection.*

Proof. Since all the members of $Mor\mathcal{C}$ are sets and since there is no maximal cardinal, it follows that, for every ordered pair $(X, Y) \in Ob\mathcal{C} \times Ob\mathcal{C}$, there are ordered pairs (A', B') and (A'', B'') of sets such that

$$card(\mathcal{C}(X, Y)) \leq card(Set(A', B')) \quad \text{and}$$
$$card(\mathcal{C}(Y, X)) \leq card(Set(B'', A'')).$$

Notice that if S, T, S', T' are sets such that $card(S) \leq card(S')$ and $card(T) \leq card(T')$, then

$$card(Set(S, T)) \leq card(Set(S', T')).$$

Consequently, for a given pair $X, Y \in Ob\mathcal{C}$, there exists a pair $A, B \in Ob(Set)$ such that

$$card(\mathcal{C}(X, Y)) \leq card(Set(A, B)) \quad \text{and}$$
$$card(\mathcal{C}(Y, X))\} \leq card(Set(B, A)).$$

Now, by assuming the axiom of choice, every set can be well ordered, and therefor, since $\mathcal{C}$ is a small category, i.e., $Ob\mathcal{C}$ is a set, there exists a well ordered set $J \equiv (J, \leq)$ such that

$$Ob\mathcal{C} = \{X_j \mid j \in J\}.$$

Further, there exists a bijective function of $Ob\mathcal{C}$ onto a subclass $\mathcal{S} \subseteq Ob(Set)$ (which is a set having the complement $Ob(Set) \setminus \mathcal{S}$ that is a (proper) class!)

$$(X_j \in Ob\mathcal{C}) \mapsto (S_j \in \mathcal{S}), \quad j \in J.$$

However, we need a very special such a subclass. In order to simplify our speech, let us say that a family $\{A_j \mid j \in J\}$ of sets A_j is *suitable, with respect to $\mathcal{C}$, for some member $A_{j'}$*, if, for each $j \in J$,

$$card(\mathcal{C}(X_{j'}, X_j)) \leq card(Set(A_{j'}, A_j)) \quad \text{and}$$
$$card(\mathcal{C}(X_j, X_{j'})) \leq card(Set(A_j, A_{j'})).$$

We are to construct such a family $\{A_j \mid j \in J\}$ of sets, i.e., a certain set $O' \subseteq Ob(Set)$, which is suitable, with respect to $\mathcal{C}$, for each its member A_j, $j \in J$. The construction is by transfinite induction. Denote $\min(J, \leq) \equiv j_0$. Firstly, there exist a family of sets $\{B_j \mid j \in J\}$ and sets B'_{j_0} and B''_{j_0} such that, for each $j \in J$,

$$card(\mathcal{C}(X_{j_0}, X_j)) \leq card(Set(B'_{j_0}, B_j)) \quad \text{and}$$
$$card(\mathcal{C}(X_j, X_{j_0})) \leq card(Set(B_j, B''_{j_0})).$$

Then by choosing any set A_{j_0} such that

$$\max\{card(B_{j_0}), card(B'_{j_0}), card(B''_{j_0})\} \leq card(A_{j_0}),$$

the family $\{A_{j_0}\} \cup \{B_j \mid j_0 < j\}$ is suitable, with respect to $\mathcal{C}$, for A_{j_0}. We proceed by transfinite construction (see [1], II.5.2). Given a $j_1 \in J$, $j_0 < j_1$ (no matter of the kind, i.e., having an immediate predecessor or being a limit one), assume that a family

$$\{A_j \mid j < j_1\} \cup \{B'_j \mid j_1 \leq j\}$$

is constructed which is suitable, with respect to $\mathcal{C}$, for each A_j, $j < j_1$. It remains to construct a set A_{j_1} and a family of sets $\{B''_j \mid j_1 < j\}$ such that the family

$$\{A_j \mid j \leq j_1\} \cup \{B''_j \mid j_1 < j\}$$

is suitable, with respect to $\mathcal{C}$, for each A_j, $j \leq j_1$. In order to do it, notice that, if C_j, $j_1 \leq j$, are sets such that $card(B'_j) \leq card(C_j)$, $j_1 \leq j$, the new family

$$\{A_j \mid j < j_1\} \cup \{C_j \mid j_1 \leq j\}$$

is still suitable, with respect to $\mathcal{C}$, for each $A_{j'}$, $j < j_1$. Clearly, there is such a C_{j_1} satisfying

$$card(\mathcal{C}(X_{j_1}, X_{j_1})) \leq card(Set(C_{j_1}, C_{j_1}))$$

Further, in addition, similarly to the basic step, there exist sets C'_{j_1} and C''_{j_1} such that the family

$$\{A_j \mid j < j_1\} \cup \{C_j \mid j_1 < j\}$$

has, for each $j \neq j_1$, the properties

$$card(\mathcal{C}(X_{j_1}, X_j)) \leq \begin{cases} card(Set(C'_{j_1}, A_j)), & j < j_1, \\ card(Set(C'_{j_1}, C_j)), & j_1 < j, \end{cases}$$

and

$$card(\mathcal{C}(X_j, X_{j_1})) \leq \begin{cases} card(Set(A_j, C''_{j_1})), & j < j_1, \\ card(Set(C_j, C''_{j_1})), & j_1 < j. \end{cases}$$

Finally, there exists a set C'''_{j_1} such that

$$\max\{card(C_{j_1}),\, card(C'_{j_1}),\, card(C''_{j_1}))\} \leq card(C'''_{j_1}).$$

Now, since $card(B'_j) \leq card(C_j)$, it is obvious that, by putting $A_{j_1} \equiv C'''_{j_1}$ and $B''_j = C_j$, $j_1 < j$, the family

$$\{A_j \mid j \leq j_1\} \cup \{B''_j \mid j_1 < j\}$$

is suitable, with respect to $\mathcal{C}$, for each A_j, $j \leq j_1$. This completes the inductive proof assuring the existence of a bijection

$$\Phi : Ob\mathcal{C} \equiv \mathcal{O} \to \mathcal{O}', \quad X_j \mapsto \Phi(X_j) = A_j, \quad j \in J.$$

and, for every ordered pair $(j, j') \in J \times J$, existence of an injection

$$\Phi^j_{j'} : \mathcal{C}(X_j, X_{j'}) \to Set(\Phi(X_j), \Phi(X_{j'})) = Set(A_j, A_{j'}).$$

Finally, for every $(j, j') \in J \times J$, choose

$$M(A_j, A_{j'}) = \Phi^j_{j'}[\mathcal{C}(X_j, X_{j'})] \subseteq Set(A_j, A_{j'}).$$

The statement now follows by Theorem 1. For the additional statement, one has also to choose (inductively) the sets A_j, $j \in J$, of increasing cardinalities. □

Remark 1. *By [2], III. 4.7, p. 25, the standard homotopy category of topological spaces, $HTop$, is not a concrete category. A simple proof of that statement may be as follows. Consider the segment $I \equiv [0,1] \subseteq \mathbb{R}$, the singleton $\{0\}$, the constant mapping $c : I \to \{0\}$ and any inclusion mapping $i_t : \{0\} \to I$, $t \in I$. Then the homotopy classes $[c]$ and $[i_t]$ are isomorphisms of $HTop$ and moreover $[i_t] = [c]^{-1}$, while $i_t c \neq 1_I$. Therefore, there is no forgetful functor of $HTop$ to Set.*

According to Remark 1, the full subcategory $H\mathcal{Q} \subseteq HTop$, determined by all closed subsets of the Hilbert cube Q, is not a concrete category. Since it is a small category, Theorems 1 and 2 imply the following fact.

Corollary 2. *The homotopy category $H\mathcal{Q}$ is isomorphic to a small category whose objects are sets and whose morphisms are functions of those sets.*

References

[1] J.Dugundji, Topology, Allyn and Bacon Inc., Boston, 1966.

[2] H. Herlich and G. E. Strecker, Category Theory, An Introduction, Allyn and Bacon Inc., Boston, 1973.

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