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Unitary $(\mathfrak{g}, K)$ modules of $SU(2,1)$

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Abstract

Let $G = SU(2,1)$. In this paper we parametrize irreducible unitary $(\mathfrak{g}, K)$ modules of G. The parametrization is done in two steps. Firstly, we parametrize irreducible $(\mathfrak{g}, K)$ modules (Theorem 10). In the second step we find unitary $(\mathfrak{g}, K)$ modules (Theorem 18). One can compare our results with [4] and [5].

Keywords: semisimple Lie groups, semisimple Lie algebras, unitary representations

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1 Introduction

Let G be a real reductive group. We will follow the definition of the real reductive group from [3]. The main goal of the representation theory is finding the unitary dual of the group G. One can approach to this problem using $(\mathfrak{g}, K)$ modules (see [1]) which correspond to admissible representations. In the first step all irreducible $(\mathfrak{g}, K)$ modules are found. In the second step it remains to find unitary $(\mathfrak{g}, K)$ modules. The first step correspond to Langlands classification. The second problem is still unsolved in general.

In this paper we find the unitary dual for the group $G = SU(2,1)$. The unitary dual of $SU(n,1)$ is already found by Kraljević (see [4] and [5]). Why should anyone solve already solved problem? We hope that our technique, which is applied for $SU(2,1)$, can be generalized and applied to some other groups. Also, the construction of unitary $(\mathfrak{g}, K)$ modules in this case is very explicit.

It is important to emphasize that for each weight $\lambda \in \mathfrak{h}^*$, given by $\lambda = (n, m)$, $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ (see Definition 2), the space of vectors spanned by v^1_{nm} is one-dimensional. It is also valid for $SU(r, 1)$ for any $r \in \mathbb{N}$, but it is not valid for $SU(r, s)$ where $r, s \in \mathbb{N}$ and $r, s > 1$. We used that fact in our construction and it simplified our calculations. Also, it is the reason why the unitary dual of $SU(r, s)$ is not known in general case.

We start with the set of coefficients $\{a_{nm}b_{nm}, c_{nm}, d_{nm}\}$ which completely describe the action of the complexified Lie algebra $\mathfrak{g}$. Actually, products ad and bc can be calculated and they give an important information about the irreducibility of $(\mathfrak{g}, K)$ modules. Also, the set of products ad and bc (explained in Theorem 8) is the key ingredient in description of unitary $(\mathfrak{g}, K)$ modules and classification of irreducible unitary $(\mathfrak{g}, K)$ modules. The main idea in this approach is to treat K types as points. The rest of construction at some points looks like a construction of irreducible unitary $(\mathfrak{g}, K)$ modules of $SL(2, \mathbb{R})$.

In Section 2 we recall basic results in representation theory of $SL(2, \mathbb{R})$. Some statements will be used later and some statements will be compared with our results. Also, we wanted to demonstrate our ideas in this case. In Section 3 we construct $(\mathfrak{g}, K)$ modules using certain set of coefficients. After that we parametrize irreducible $(\mathfrak{g}, K)$ modules of $G = SU(2, 1)$. The parametrization is given in Theorem 10. In Section 4 we parametrize unitary $(\mathfrak{g}, K)$ modules. The key step is done in Theorem 16. The parametrization is given in Theorem 18.

Lie groups will be denoted by capital letters, corresponding Lie algebras by Gothic letters with subscript 0 and complexified Lie algebras by Gothic letters without subscript. For example, the Lie algebra of G will be denoted by $\mathfrak{g}_0$ and the complexified Lie algebra by $\mathfrak{g}$. If H, X and Y span a basis for $\mathfrak{sl}(2,\mathbb{C})$ such that $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$ then we say that $\mathfrak{sl}(2,\mathbb{C})$ is represented by a triple (H, X, Y). If π, V is the representation of $\mathfrak{g}$, usually, we will write $X.v$ instead of $\pi(X)v$ for $X \in \mathfrak{g}$ and $v \in V$.

2 Unitary dual of $SL(2, \mathbb{R})$

Let $G = SL(2, \mathbb{R})$, $K = SO(2)$, $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{k} = \mathfrak{so}(2)$. Unitary representations of G are well known, see [2]. However, we redo the construction since some details appear later.

We use notation and results from [6]. The basis of $\mathfrak{g}$ contains elements

$$H = -i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$X = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \frac{1}{2}(A + iB)$$

and

$$Y = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \frac{1}{2}(A - iB).$$

It is easy to check that

$$[iH, B] = 2A, \quad [A, iH] = 2B \quad \text{and} \quad [B, A] = -2iH.$$

Let W be a $\mathfrak{s}l(2,\mathbb{C})$ module. Then we can choose a basis $\{w^k \mid k \in S \subset \mathbb{Z}\}$ of W such that $w^k \in W$, $H.w^k = kw^k$ and

$$X.w^k = \frac{1}{2}(\lambda + (k+1))\, w^{k+2} = a_k w^{k+2}$$

and

$$Y.w^k = \frac{1}{2}(\lambda - (k-1))\, w^{k-2} = b_k w^{k-2}$$

for some $\lambda \in \mathbb{C}$ and some set S ([6], Lemma 1.2.6). In [6], they analyze irreducible $(\mathfrak{g},K)$ modules and do not mention the set S. Here, at this point, we concentrate on coefficients a_k and b_k. The set S can have the form $\{2m \mid m \in \mathbb{Z}\}$ or $\{1+2m \mid m \in \mathbb{Z}\}$. If $\lambda \notin \mathbb{Z}$, then the module W is irreducible. If $\lambda \in \mathbb{Z}$, then the module W has submodules. It is easy to see that

$$A.w^k = (X + Y).w^k = a_k w^{k+2} + b_k w^{k-2}$$

and

$$B.w^k = i(-X + Y).w^k = i\left(-a_k w^{k+2} + b_k w^{k-2}\right).$$

If we consider a finite-dimensional module V of dimension n, we will use the same basis, with different indexes denoted by v, such that $H.v^k = (n+1-2k)v^k$ and

$$X.v^k = -(k-1)v^{k-1} \tag{1}$$

and

$$Y.v^k = -(n-k)v^{k+1}. \tag{2}$$

We can assume that $v^0 = v^{n+1} = 0$.

Let us determine λs for which it is possible to construct an inner product $\langle \cdot, \cdot \rangle \colon W \times W \to \mathbb{C}$ such that (26) is satisfied. Let us assume that $\langle w^k, w^l \rangle \neq 0$ for some k and l. Then,

$$\langle iH.w^k, w^l \rangle = \langle w^k, (iH)^*.w^l \rangle$$

and (26) show that $ik\langle w^k, w^l\rangle = il\langle w^k, w^l\rangle$. Hence

$$\langle w^k, w^l\rangle = 0 \quad \text{for all } k \neq l. \tag{3}$$

Now, from (3), it follows that

$$A^*.w^k = \overline{b_{k+2}}\frac{\|w^k\|^2}{\|w^{k+2}\|^2}w^{k+2} + \overline{a_{k-2}}\frac{\|w^k\|^2}{\|w^{k-2}\|^2}w^{k-2}$$

and

$$B^*.w^k = i\left(-\overline{b_{k+2}}\frac{\|w^k\|^2}{\|w^{k+2}\|^2}w^{k+2} + \overline{a_{k-2}}\frac{\|w^k\|^2}{\|w^{k-2}\|^2}w^{k-2}\right).$$

Each time we get the same condition:

$$a_k + \overline{b_{k+2}}\frac{\|w^k\|^2}{\|w^{k+2}\|^2} = 0, \quad \forall k.$$

or

$$\frac{a_k}{\overline{b_{k+2}}} = -\frac{\|w^k\|^2}{\|w^{k+2}\|^2} \in (-\infty, 0)\,, \quad \forall k.$$

However, we prefer to multiply it by $b_{k+2}\overline{b_{k+2}} = \|b_{k+2}\|^2$ and say that the irreducible representation is unitary if and only if

$$a_k b_{k+2} \in (-\infty, 0), \quad \forall k. \tag{4}$$

We consider an open interval $(-\infty, 0)$ since the case $a_k b_{k+2} = 0$ leads to reducibility. It will be explained in Remark 11. Relation (4) transforms to

$$(\lambda + k + 1)(\lambda - (k+1)) = \lambda^2 - (k+1)^2 \in (-\infty, 0)\,, \quad \forall k.$$

Let us recall that $k = 2z$ or $k = 2z + 1$ for $z \in \mathbb{Z}$. If $k = 2z$, λ can be equal to ri for $r \in \mathbb{R}$ (it corresponds to principal series), $r \in (-1, 1)$ (it corresponds to complementary series) and $2m+1$ for $m \in \mathbb{Z}$ (submodules correspond to discrete seris and the trivial representation). If $k = 2z+1$, λ can be equal to ri for $r \in \mathbb{R}^*$ (it corresponds to prinicipal series), 0 (submodules correspond to mock discrete series) and $2m$ for $m \in \mathbb{Z}$ (submodules correspond to discrete series).

It remains to analyze (unitary) finite-dimensional representations of the compact real form of $\mathfrak{g}$. For the beginning, let us choose the basis:

$$-\frac{1}{2}iH, \quad \frac{1}{2}(X - Y) = \frac{1}{2}iB \quad \text{and} \quad -\frac{1}{2}i(X + Y) = -\frac{1}{2}iA.$$

Then $-\frac{1}{2}iH.v^k = -\frac{1}{2}i(n+1-2k)v^k$,

$$\frac{1}{2}(X-Y).v^k = \frac{1}{2}\left(-(k-1)v^{k-1} + (n-k)v^{k+1}\right)$$

and

$$-\frac{1}{2}i(X+Y).v^k = \frac{1}{2}i\left((k-1)v^{k-1} + (n-k)v^{k+1}\right).$$

Element $-\frac{1}{2}iH$ satisfies (26) (for any inner product satisfying (3)). It remains to analyze remaining two elements. In any case, (26) produces the same condition:

$$\|v^{k+1}\|^2 = \frac{k}{n-k}\|v^k\|^2. \tag{5}$$

It will be useful to express $\|v^k\|^2$ in terms of $\|v^1\|^2$. Using induction, one can easily show that

$$\|v^k\|^2 = \frac{(k-1)!(n-k)!}{(n-1)!}\|v^1\|^2 = \frac{1}{\binom{n-1}{k-1}}\|v^1\|^2. \tag{6}$$

Example 1. *Let V be the $\mathfrak{su}(2)$ module such that* $\dim V = 5$ *and* $\|v^1\| = 1$. *Then*

$$\|v^2\| = \frac{1}{2}, \quad \|v^3\| = \frac{1}{\sqrt{6}}, \quad \|v^4\| = \frac{1}{2}, \quad \|v^5\| = 1.$$

3 $(\mathfrak{g}, K)$ modules for $SU(2,1)$

Let $G = SU(2,1)$. Then $K = S(U(2)\times U(1)) = SU(2)\times S^1$, $\mathfrak{k}_0 = \mathfrak{su}(2)\oplus\mathbb{R}$ and $\mathfrak{k} = \mathfrak{sl}(2,\mathbb{C})\oplus\mathbb{C}$. This $\mathfrak{sl}(2,\mathbb{C})$ can be represented by a triple $(H_\alpha, X_\alpha, Y_\alpha)$. It remains to set $\mathbb{C} = Z\mathbb{C}$ where $Z = H_\alpha + 2H_\beta$. Let us define a basis for $\mathfrak{g}$. The Cartan subalgebra $\mathfrak{h}\subset\mathfrak{k}$ is generated by

$$H_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad H_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Now, we define

$$X_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } X_{\alpha+\beta} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Elements Y_α, Y_β and $Y_{\alpha+\beta}$ are defined similarly. The next step is to define a basis for $\mathfrak{g}_0$. We take iH_α and iH_β for the Cartan subalgebra and

$$A_\alpha = X_\alpha - Y_\alpha \quad \text{and} \quad B_\alpha = i(X_\alpha + Y_\alpha)$$

for the remainder of $\mathfrak{k}_0$. The rest of $\mathfrak{g}_0$ is given by

$$\begin{aligned} A_\beta &= X_\beta + Y_\beta, & B_\beta &= i(X_\beta - Y_\beta) \\ A_{\alpha+\beta} &= X_{\alpha+\beta} + Y_{\alpha+\beta} \quad \text{and} & B_{\alpha+\beta} &= i(X_{\alpha+\beta} - Y_{\alpha+\beta}). \end{aligned} \tag{7}$$

One should notice a different sign in expressions for α and β.

Let us consider irreducible representations of K. Since $\mathfrak{k} = \mathfrak{s}l(2,\mathbb{C}) \oplus \mathbb{C}Z$, where $Z = H_\alpha + 2H_\beta$, irreducible representations of $\mathfrak{k}$ are irreducible representations of $\mathfrak{s}l(2,\mathbb{C})$ on which Z acts as a multiplication by scalars. Since $\mathbb{C}$Z ($\mathbb{R}iZ$) corresponds to a circle (K is compact), the scalar m is an integer.

Definition 2. *We denote K modules by V_{nm} where n is the dimension of the space V_{nm} and m is the scalar by which $Z = H_\alpha + 2H_\beta$ acts on that space. Let $\{v^k_{nm}\}$, $k \in \{1, 2, \ldots, n\}$ be a basis for V_{nm} defined above ($H_\alpha.v^k = (n + 1 - 2k)v^k$, $X.v^k = -(k - 1)v^{k-1}$ (as in (1))) and $Y.v^k = -(n - k)v^{k+1}$).*

Hence, $Z.v^k_{nm} = (H_\alpha + 2H_\beta).v^k_{nm} = mv^k_{nm}$ for any element $v^k_{nm} \in V_{nm}$. Let V be an irreducible $(\mathfrak{g}, K)$ module. Then the restriction of V to K has the form

$$V|_K = \bigoplus_{n\in\mathbb{N},\, m\in\mathbb{Z}} V_{nm}.$$

It should be more correct to write $(n, m) \in S(V)$ where $S(V) \subset \mathbb{N} \times \mathbb{Z}$ is some set which depends on V. For example, it is easy to see that $n + m$ is an odd number. However, we want to emphasize that n in the natural number and m is an integer. This set $S(V)$ will be discussed later (after the following theorem which describes relationship among K types). Now, it is important to emphasize that we know that the multiplicity of K modules V_{nm} is 1 or, equivalently, the dimension of the space spanned by the vectors v^1_{nm} is 1 for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. It makes this construction possible. If our group G is more complicated ($SU(2, 2)$), multiplicities are bigger then 1 and the construction is more complex. The author currently works on this problem.

Theorem 3. *Let V be an irreducible $(\mathfrak{g}, K)$ module and let $\{v^k_{nm}\}$ be a basis for V_{nm} as in Definition 2. Then*

$$X_{\alpha+\beta}.v^k_{nm} = a_{nm}v^k_{n+1\,m+3} + \frac{k-1}{n-1}c_{nm}v^{k-1}_{n-1\,m+3}, \tag{8}$$

$$\begin{aligned} X_\beta.v^k_{nm} &= -a_{nm}v^{k+1}_{n+1\,m+3} + \frac{n-k}{n-1}c_{nm}v^k_{n-1\,m+3}, \\ Y_{\alpha+\beta}.v^k_{nm} &= b_{nm}v^{k+1}_{n+1\,m-3} + \frac{n-k}{n-1}d_{nm}v^k_{n-1\,m-3}, \\ Y_\beta.v^k_{nm} &= b_{nm}v^k_{n+1\,m-3} - \frac{k-1}{n-1}d_{nm}v^{k-1}_{n-1\,m-3}. \end{aligned} \tag{9}$$

for some coefficients a_{nm}*,* b_{nm}*,* c_{nm} *and* d_{nm}*.*

Remark 4. *Possible values of* n *and* m *in* V *will be determined gradually later. At this moment we can say that the set* $\{v^1_{nm}\}$ *looks like a cone (or a subset of a cone). Also, one should notice that if* V_{nm} *is a* K *type of some* $(\mathfrak{g}, K)$ *module* V*, then possible* K *types of* V *for fixed* n *and have the form* $V_{n\,m+6k}$ *for* $k \in \mathbb{Z}$ *and* K *types of the form* $V_{n\,m+6k+2}$ *and* $V_{n\,m+6k+4}$ *belong to some other* $(\mathfrak{g}, K)$ *modules which have no common* K *types with* V*. Also,* $V_{n\,m+2k+1}$*,* $k \in \mathbb{Z}$*, are never* K *types for any* $(\mathfrak{g}, K)$ *module.*

Remark 5. *One can ask if the value of* n *can be equal to 1 in* (8) *and* (9) *since the denominator of some fractions is* $n-1$*. Let us analyze the statement of the theorem. The action of* $X_{\alpha+\beta}$ *and* X_β *on vectors in* V_{nm} *produces vectors which are sums of vectors from* $V_{n+1\,m+3}$ *and* $V_{n-1\,m+3}$*. It is one of main ideas of this paper: the action is not "wild", it can be understood completely. If* $n = 1$*, the space* $V_{0\,m-3}$ *does not exist (the dimension of that space is 0) and it should be more correct to write* $X_{\alpha+\beta}.v^k_{1m} = a_{nm}v^k_{2\,m+3}$ *and* $X_\beta.v^k_{1m} = -a_{nm}v^{k+1}_{2\,m+3}$ *instead of* (8)*. However, we did not want to write it as separate statements. Similar statements are valid for* $Y_{\alpha+\beta}$ *and* Y_β*. Finally,* c_{1m} *and* d_{1m} *will be 0 in our calculations.*

Remark 6. *This theorem shows that the set* $\{V_{nm}\}$ *and coefficients* a_{nm}*,* b_{nm}*,* c_{nm} *and* d_{nm} *determine the structure of the* $(\mathfrak{g}, K)$ *modules* V*. It is clear that these coefficients are not determined uniquely. However, the products* $a_{nm}d_{n+1\,m+3}$ *and* $b_{nm}c_{n+1\,m-3}$ *are unique and it is an important observation.*

Remark 7. *One can say that irreducible* K *modules* V_{nm} *can be represented as points. It is the main idea in our construction. We want to understand the structure of* K *modules.*

Proof. We will prove the first two relations. Let us consider $X_{\alpha+\beta}.v^1_{nm}$. It is clear that

$$X_{\alpha+\beta}.v^1_{nm} \in \bigoplus_{p\geq n+1} V_{p\,m+3}.$$

Let us assume that $X_{\alpha+\beta}.v^1_{nm} = a + b$ for $a \in V_{q\,m+3}$, where $q > n+1$ and $b \in \bigoplus_{p\geq n+1,\,p\neq q} V_{p\,m+3}$. Then $X_\alpha X_{\alpha+\beta}.v^1_{nm} \neq 0$. It produces a contradiction since $X_{\alpha+\beta}X_\alpha.v^1_{nm} = 0$ and $[X_\alpha, X_{\alpha+\beta}] = 0$. It shows that

$$X_{\alpha+\beta}.v^1_{nm} = a_{nm}v^1_{n+1\,m+3} \tag{10}$$

for some coefficient a_{nm}. Now, let us calculate $X_\beta v^1_{nm}$. Similar calculation shows that

$$X_\beta.v^1_{nm} = \lambda v^2_{n+1\,m+3} + c_{nm}v^1_{n-1\,m+3}.$$

for some coefficients λ and c_{nm}. The action of X_α and (1) produces

$$X_\alpha X_\beta.v^1_{nm} = -\lambda v^1_{n+1\,m+3}.$$

Since $[X_\alpha, X_\beta] = X_{\alpha+\beta}$ and $X_\alpha.v^1_{nm} = 0$,

$$X_{\alpha+\beta}.v^1_{nm} = [X_\alpha, X_\beta].v^1_{nm} = -\lambda v^1_{n+1\,m+3}. \tag{11}$$

Now, (10) and (11) show that $\lambda = -a_{nm}$.

We continue by induction on k. The base of induction is just proved. Let us assume that first two relations are valid for k. Since $[Y_\alpha, X_\beta] = 0$,

$$\begin{aligned}
X_\beta.v^{k+1}_{nm} &= -\frac{1}{n-k}X_\beta Y_\alpha.v^k_{nm} = -\frac{1}{n-k}Y_\alpha X_\beta.v^k_{nm} \\
&= -\frac{1}{n-k}Y_\alpha.\left(-a_{nm}v^{k+1}_{n+1\,m+3} + \frac{n-k}{n-1}c_{nm}v^k_{n-1\,m+3}\right) \\
&= \frac{a_{nm}}{n-k}(-(n-k))v^{k+2}_{n+1\,m+3} - \frac{c_{nm}}{n-1}(-(n-1-k))v^{k+1}_{n-1\,m+3} \\
&= -a_{nm}v^{k+2}_{n+1\,m+3} + \frac{n-(k+1)}{n-1}c_{nm}v^{k+1}_{n-1\,m+3}.
\end{aligned}$$

Now we use this result and obtain

$$\begin{aligned}
X_{\alpha+\beta}.v^{k+1}_{nm} &= (X_\alpha X_\beta - X_\beta X_\alpha).v^{k+1}_{nm} \\
= X_\alpha.\left(-a_{nm}v^{k+2}_{n+1\,m+3} + \frac{n-(k+1)}{n-1}c_{nm}v^{k+1}_{n-1\,m+3}\right) &- X_\beta.\left((-k)v^k_{nm}\right) \\
&= a_{nm}(k+1)v^{k+1}_{n+1\,m+3} - \frac{n-(k+1)}{n-1}kc_{nm}v^k_{n-1\,m+3} \\
&\quad + k\left(-a_{nm}v^{k+1}_{n+1\,m+3} + \frac{n-k}{n-1}c_{nm}v^k_{n-1\,m+3}\right) \\
&= a_{nm}v^{k+1}_{n+1\,m+3} + k\left(-\frac{n-(k+1)}{n-1} + \frac{n-k}{n-1}\right)c_{nm}v^k_{n-1\,m+3} \\
&= a_{nm}v^{k+1}_{n+1\,m+3} + \frac{k}{n-1}c_{nm}v^k_{n-1\,m+3}.
\end{aligned}$$

Remaining two relations can be proved similarly. □

Theorem 8. *Let V be an irreducible $(\mathfrak{g}, K)$ module. Then coefficients a_{nm}, b_{nm}, c_{nm} and d_{nm} satisfy following relations,*

$$-\frac{1}{n}a_{nm}d_{n+1\,m+3}+b_{nm}c_{n+1\,m-3}-c_{nm}b_{n-1\,m+3}=\frac{m-n+1}{2}, \quad (12)$$

$$-a_{nm}d_{n+1\,m+3}+\frac{1}{n}b_{nm}c_{n+1\,m-3}+d_{nm}a_{n-1\,m-3}=\frac{m+n-1}{2}, \quad (13)$$

$$a_{nm}d_{n+1\,m+3}=d_{nm}a_{n-1\,m-3}=b_{nm}c_{n+1\,m-3}=c_{nm}b_{n-1\,m-3}=0$$
$$\text{when } V_{nm} \text{ is } K \text{ type and } V_{n\pm1\,m\pm3} \text{ is not } K \text{ type} \quad (14)$$

$$b_{nm}a_{n+1\,m-3}=a_{nm}b_{n+1\,m+3}, \quad (15)$$

$$d_{nm}c_{n-1\,m-3}=c_{nm}d_{n-1\,m+3}, \quad (16)$$

$$(n+1)a_{nm}c_{n+1\,m+3}=nc_{nm}a_{n-1\,m+3}, \quad (17)$$

$$(n+1)b_{nm}d_{n+1\,m-3}=nd_{nm}b_{n-1\,m-3}. \quad (18)$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ for which V_{nm} is K type of V. If the set of K types is given together with relations (12) – (18)*, then it is possible to reconstruct an irreducible $(\mathfrak{g}, K)$ module V.*

Remark 9. *The theorem enumerates necessary and sufficient conditions for the existence of (irreducible) $(\mathfrak{g}, K)$ module V.*

Proof. Let us assume that $(\mathfrak{g}, K)$ module exists. Let us consider a nonzero element v^k_{nm} and apply the relation $[X_\beta, Y_\beta] = H_\beta$ on that element. The left hand side is equal to

$$\begin{aligned}
&(X_\beta Y_\beta - Y_\beta X_\beta).v^k_{nm}\\
=&X_\beta.\left(b_{nm}v^k_{n+1\,m-3}-\frac{k-1}{n-1}d_{nm}v^{k-1}_{n-1\,m-3}\right)\\
&-Y_\beta.\left(-a_{nm}v^{k+1}_{n+1\,m+3}+\frac{n-k}{n-1}c_{nm}v^k_{n-1\,m+3}\right)\\
=&b_{nm}\left(-a_{n+1\,m-3}v^{k+1}_{n+2\,m}+\frac{n+1-k}{n}c_{n+1\,m-3}v^k_{nm}\right)\\
&-\frac{k-1}{n-1}d_{nm}\left(-a_{n-1\,m-3}v^k_{nm}+\frac{n-k}{n-2}c_{n-1\,m-3}v^{k-1}_{n-2\,m}\right)\\
&+a_{nm}\left(b_{n+1\,m+3}v^{k+1}_{n+2\,m}-\frac{k}{n}d_{n+1\,m+3}v^k_{nm}\right)\\
&-\frac{n-k}{n-1}c_{nm}\left(b_{n-1\,m+3}v^k_{nm}-\frac{k-1}{n-2}d_{n-1\,m+3}v^{k-1}_{n-2\,m}\right).
\end{aligned}$$

The right hand side is equal to

$$H_\beta.v^k_{nm}=\frac{m-n-1+2k}{2}v^k_{nm}.$$

One can compare coefficients of $v^{k-1}_{n-2\,m}$ and obtain (16). Similarly, coefficient of $v^{k+1}_{n+2\,m}$ produces (15). Finally, coefficient of v^k_{nm} produces

$$-\frac{k}{n}a_{nm}d_{n+1\;m+3}+\frac{n+1-k}{n}b_{nm}c_{n+1\,m-3}$$
$$-\frac{n-k}{n-1}c_{nm}b_{n-1\,m+3}+\frac{k-1}{n-1}d_{nm}a_{n-1\,m-3}=\frac{m-n-1+2k}{2}. \quad (19)$$

Also, $a_{nm}d_{n+1\;m+3}=0$ if $V_{n+1\,m+3}$ is not K type of V, $b_{nm}c_{n+1\;m-3}=0$ if $V_{n+1\,m-3}$ is not K type of V, $c_{nm}b_{n-1\;m+3}=0$ if $V_{n-1\,m+3}$ is not K type of V and $d_{nm}a_{n-1\;m-3}=0$ if $V_{n-1\,m-3}$ is not K type of V and it is (14). For $k=1$, one obtains (12) and for $k=n$ it transforms to (13). It is easy to check that (19) is a linear combination of (12) and (13), namely

$$\frac{n-k}{n-1}(12)+\frac{k-1}{n-1}(13)=(19).$$

It shows that it is enough to consider (12) and (13). These two relations are more convenient then (19) since k does not appear in (12) and (13).

If we apply the relation $X_\beta X_{\alpha+\beta}=X_{\alpha+\beta}X_\beta$ on the element v^k_{nm} we obtain (17). Finally, if we apply the relation $Y_\beta Y_{\alpha+\beta}=Y_{\alpha+\beta}Y_\beta$ on the element v^k_{nm} we obtain (18). One can check all other commutation relations in $\mathfrak{g}$, but it will not produce new conditions on coefficients a_{nm}, b_{nm}, c_{nm} and d_{nm}. It shows that (12) – (18) have to be satisfied.

Now, let us assume that the set of K types together with coefficients a_{nm}, b_{nm}, c_{nm} and d_{nm} are given and (12) – (18) are satisfied. At the beginning of the proof of the Theorem 10, we will show that K types (or vectors v^1_{nm}) form a cone or a subset of a cone (a strip or a parallelogram). The proof is technical and follows directly from (12) – (18). Also, a_{nm}, b_{nm}, c_{nm} and d_{mm} are different from 0 if V_{nm} and $V_{n\pm1\,m\pm3}$ are K types of an irreducible $(\mathfrak{g},K)$ module V. Let us choose any K type V_{nm} of V and $v^1_{nm}\in V_{nm}$. Then vectors v^k_{nm} are defined by (2), where Y in (2) is Y_α. Vectors $v^1_{n+1\,m\pm3}$ are defined by $v^1_{n+1\,m+3}=\frac{1}{a_{nm}}X_{\alpha+\beta}.v^1_{nm}$ (by (8)) and $v^1_{n+1\,m-3}=\frac{1}{b_{nm}}Y_\beta.v^1_{nm}$. Vectors $v^k_{n+1\,m\pm3}$ are defined by (2) (again Y in (2) is Y_α). Vectors $v^1_{n-1\,m\pm3}$ are defined by $v^1_{n-1\,m+3}=\frac{1}{c_{nm}}(X_\beta.v^1_{nm}+a_{nm}v^2_{n+1\,m+3})$ and $v^1_{n-1\,m-3}=\frac{1}{d_{nm}}(Y_{\alpha+\beta}.v^1_{nm}-b_{nm}v^2_{n+1\,m-3})$. Again, vectors $v^k_{n-1\,m\pm3}$ are defined by (2). Since, the structure of K types is simple, all K types can be reached in this way. One can ask if this definition is good, or equivalently, is it possible to get two different values for the same vector. It is not possible since all commutation relations are satisfied. For example $v^1_{n+2\,m}$ can be obtained as

$$v^1_{n+2\,m}=\frac{1}{a_{n+1\,m-3}}X_{\alpha+\beta}.\frac{1}{b_{nm}}Y_\beta.v^1_{nm}$$

and

$$v^1_{n+2\,m} = \frac{1}{b_{n+1\,m+3}} Y_\beta . \frac{1}{a_{nm}} X_{\alpha+\beta} . v^1_{nm}$$

The commutator of $X_{\alpha+\beta}$ and Y_β is X_α, but $X_\alpha . v^1_{nm} = 0$. Also $a_{n+1\,m-3} b_{nm} = b_{n+1\,m+3} a_{nm}$ by (15). We can also check it for $k > 1$. If we start with the definition of $v^1_{n+1\,m+3}$,

$$v^1_{n+1\,m+3} = \frac{1}{a_{nm}} X_{\alpha+\beta} . v^1_{nm}$$

and act by Y_α on it, we obtain

$$Y_\alpha . v^1_{n+1\,m+3} = \frac{1}{a_{nm}} \left(X_{\alpha+\beta} Y_\alpha + X_\beta \right) . v^1_{nm}.$$

One can apply formulas from Theorem 3 and get equality. However, using these formulas will repeat the proof and give the spirit of that theorem. □

It is more convenient to work with $a_{nm} d_{n+1\,m+3}$ and $b_{nm} c_{n+1\,m-3}$ then a_{nm}, b_{nm}, c_{nm} and d_{nm}. The reason is very simple. The later expressions are not determined uniquely since they depend on the choice of vectors v^k_{nm}. Hence, we plan to determine expressions $a_{nm} d_{n+1\,m+3}$ and $b_{nm} c_{n+1\,m-3}$ using (12) and (13) and then show that it is possible to determine coefficients a_{nm}, b_{nm}, c_{nm} and d_{nm} such that all relations above are satisfied (and given $(\mathfrak{g}, K)$ module exists). We will be able to give explicit formulas for expressions $a_{nm} d_{n+1\,m+3}$ and $b_{nm} c_{n+1\,m-3}$. Then, it is easy to give formulas for coefficients a_{nm}, b_{nm}, c_{nm} and d_{nm}.

Let us write (18) for $m + 6$ instead of m and multiply by (17). It produces

$$\begin{aligned}(n+1)^2 a_{nm} d_{n+1\,m+3} b_{n\,m+6} c_{n+1\,m+3} \\ = n^2 a_{n-1\,m+3} d_{n\,m+6} b_{n-1\,m+3} c_{nm}\end{aligned} \tag{20}$$

and

$$\frac{a_{nm} d_{n+1\,m+3}}{a_{n-1\,m+3} d_{n\,m+6}} \frac{b_{n\,m+6} c_{n+1\,m+3}}{b_{n-1\,m+3} c_{nm}} = \frac{n^2}{(n+1)^2}. \tag{21}$$

Now, (15) and (16) show that $\dfrac{a_{nm}}{a_{n-1\,m+3}} = \dfrac{b_{n\,m+6}}{b_{n-1\,m+3}}$ and $\dfrac{d_{n+1\,m+3}}{d_{n\,m+6}} = \dfrac{c_{n+1\,m+3}}{c_{nm}}$. We conclude that (21) transforms to

$$\left(\frac{a_{nm} d_{n+1\,m+3}}{a_{n-1\,m+3} d_{n\,m+6}} \right)^2 = \frac{n^2}{(n+1)^2}$$

or

$$\frac{a_{nm}d_{n+1\,m+3}}{a_{n-1\,m+3}d_{n\,m+6}} = \frac{b_{n\,m+6}c_{n+1\,m+3}}{b_{n-1\,m+3}c_{nm}} = \pm\frac{n}{n+1}.$$

It is possible to give more precise statement. Using induction, one can obtain

$$\frac{a_{nm}d_{n+1\,m+3}}{a_{n-1\,m+3}d_{n\,m+6}} = \frac{b_{n\,m+6}c_{n+1\,m+3}}{b_{n-1\,m+3}c_{nm}} = \frac{n}{n+1}.$$

This relation will be a consequence of formulas (22) and (23). However, we mention it now in order to give a better insight into the structure of coefficients a_{nm}, b_{nm}, c_{nm} and d_{nm}.

Theorem 10. *For any $c \in \mathbb{C}$ and $t \in \mathbb{Z}$ there exist a $(\mathfrak{g}, K)$ module $V(c, 2t)$ such that*

$$V(c,2t)|_K = V_{1\,2t} \oplus \bigoplus_{n,m\in\mathbb{Z},\, n>1} V_{nm}$$

and $a_{1\,2t}d_{2\,2t+3} = c - \frac{1}{2}t$. This module can be reducible. Any other irreducible $(\mathfrak{g}, K)$ module V is a submodule, quotient or subquotient of some $V(c, 2t)$.

Remark 11. *We are concentrated on irreducible $(\mathfrak{g}, K)$ modules. Reducibility of modules $V(c, 2t)$ will be obtained when some product(s) $a_{nm}d_{n+1\,m+3}$ or $b_{nm}c_{n+1\,m-3}$ are equal to 0. The theorem says that irreducible $(\mathfrak{g}, K)$ modules can be obtained a submodules, quotients or subquotients. Once we have formulas (24) – (25), it will be possible to determine if we have a submodule, quotient or subquotient. However, it will not be important for us. We are looking for irreducible $(\mathfrak{g}, K)$ modules and their description. Finally, it is, maybe, possible to find another choice of coefficients a_{nm}, b_{nm}, c_{nm} and d_{nm} in (24) – (25) and it would lead to another relationship among our modules. Hence, by abuse of notation, we will say just a submodule.*

Proof. Let us put $a_{1\,2t}d_{2\,2t+3} = c - \frac{1}{2}t$. Then (12) (and also (13)) shows that

$$b_{1\,2t}c_{2\,2t-3} = c + \frac{1}{2}t.$$

If $a_{1+k\,2t+3k}d_{2+k\,2t+3+3k} \neq 0$, then $b_{1+k\,2t+6+3k}c_{2+k\,2t+3+3k} = 0$ for $k \geq 0$ (by 20). If $b_{1+k\,2t-3k}c_{2+k\,2t-3-3k} \neq 0$, then $a_{1+k\,2t-6-3k}a_{2+k\,2t-3-3k} = 0$ for $k \geq 0$. It means that K modules of the irreducible component of $V(c, 2t)$ can form a cone (if $a_{1+k\,2t+3k}d_{2+k\,2t+3+3k} \neq 0$ and $b_{1+k\,2t-3k}c_{2+k\,2t-3-3k} \neq 0$ for $k \geq 0$), a strip (if $a_{1+k\,2t+3k}d_{2+k\,2t+3+3k} = 0$ for some $k \in \mathbb{N}\cup\{0\}$ or $b_{1+k\,2t-3k}c_{2+k\,2t-3-3k} = 0$ for some $k \in \mathbb{N}\cup\{0\}$) or a parallelogram (if $a_{1+k\,2t+3k}d_{2+k\,2t+3+3k} = 0$ for some $k \in \mathbb{N} \cup \{0\}$ and $b_{1+l\,2t-3l}c_{2+l\,2t-3-3l} = 0$ for some $l \in \mathbb{N} \cup \{0\}$).

We claim that our expressions ad and bc are determined uniquely. Relations (12) and (13), for $n > 1$, produce two independent equations. It is enough to walk from one vertex to another where two expressions ad and bc are already determined and calculate remaining two. A reader can easily reconstruct the path. Formulas for the vertex which is obtained by moving p steps in the $\alpha + \beta$ direction and q steps in $-\beta$ direction, have the form

$$a_{1+p+q\,2t+3p-3q} d_{2+p+q\,2t+3p-3q+3} = \frac{p+1}{p+q+2}\left(2c - (p+1)t - p(p+2)\right) \tag{22}$$

and

$$b_{1+p+q\,2t+3p-3q} c_{2+p+q\,2t+3p-3q-3} = \frac{q+1}{p+q+2}\left(2c + (q+1)t - q(q+2)\right). \tag{23}$$

They can be checked directly. One could write $n+1$ instead of $p+q+2$ in denominators of (22) and (23).

Coefficients a_{nm}, b_{nm}, c_{nm} and d_{nm} can be defined by

$$\begin{aligned} a_{1+p+q\,2t+3p-3q} &= 2c - (p+1)t - p(p+2), & (24)\\ b_{1+p+q\,2t+3p-3q} &= 2c + (q+1)t - q(q+2), \\ c_{2+p+q\,2t+3p-3q-3} &= \frac{q+1}{p+q+2}, \\ d_{2+p+q\,2t+3p-3q+3} &= \frac{p+1}{p+q+2}. & (25) \end{aligned}$$

One can check that (15) – (18) are satisfied. By Theorem 8, $(\mathfrak{g}, K)$ module $V(c, 2t)$ is well defined.

Now, it remains to show that any other module is a submodule (see Remark 11) of some $V(c, 2t)$. Let us consider some irreducible $(\mathfrak{g}, K)$ module $W(r, s)$, $r \in \mathbb{N}$, $r > 1$ and $s \in \mathbb{Z}$ of the form

$$W(r,s)|_K = V_{rs} \oplus \bigoplus_{n,m\in\mathbb{Z},\, n>r} V_{nm}$$

Let us notice that $r + s = 2z + 1$ for some $z \in \mathbb{Z}$. This time (since $r > 1$), the system of two equations produced by (12) and (13) has a unique solution. It shows that coefficients ad and bc are uniquely determined for the fixed choice of r and s. It remains to show that $W(r, s)$ is a submodule of some $V(c, 2t)$ but it is straightforward: $W(r, s)$ is submodule of

$$V\left(\frac{(r-1)(-r+1+s)-2}{4}, -3r+3+s\right)$$

and

$$V\left(\frac{(r-1)(-r+1-s)-2}{4}, 3r-3+s\right).$$

Since $r+s=2z+1$, $-3r+3+s=2z-4r+4$. $\square$

Example 12. *Let us consider the module* $W(4,3)$. *It is a submodule of* $V\left(-\frac{1}{2},-6\right)$ *and* $V(-5,12)$. *It is a nice exercise to calculate products* $a_{nm}d_{n+1\,m+3}$ *and* $b_{nm}c_{n+1\,m-3}$ *for* $W(4,3)$ *using* (12) *and* (13) *(solving system for each vertex) and compare with* (22) *and* (23) *for* $V\left(-\frac{1}{2},-6\right)$ *and* $V(-5,12)$.

4 Unitary dual of $SU(2,1)$

For the beginning we give a definition of unitary $(\mathfrak{g},K)$ modules for any real reductive group.

Definition 13. *Unitary* $(\mathfrak{g},K)$ *module* V *is a* $(\mathfrak{g},K)$ *module equipped with the inner product* $\langle\cdot,\cdot\rangle\to\mathbb{C}$ *such that*

$$(X^*+X).v=0, \quad \forall X\in\mathfrak{g}_0,\ \forall v\in V. \tag{26}$$

and the action of K *is unitary.*

Remark 14. *Since* K *acts on finite-dimensional spaces, the action is automatically unitary on* K_0. *Hence, we have to check that the action is unitary only for some representatives of connected components. Since the group* $G=SU(2,1)$ *is connected, it remains to check only* (26).

Now, let us construct the inner product mentioned in Definition 13. Using the same way of reasoning as we did for (3), one can conclude that

$$\langle v_{nm}^k, v_{rs}^l\rangle=0, \quad v_{nm}^k\in V_{nm},\ v_{rs}^l\in V_{rs}$$

for $s\neq m$ or $r-2l\neq n-2k$. It remains to consider the situation when $s=m$ and $r-2l=n-2k$.

Lemma 15. *Let* V *be a unitary* $(\mathfrak{g},K)$ *module,* $v_{nm}^k\in V_{nm}$ *and* $v_{n-2k+2l\,m}^l\in V_{n-2k+2l\,m}$. *Then*

$$\langle v_{nm}^k, v_{n-2k+2l\,m}^l\rangle\neq 0$$

if and only if $k=l$.

Proof. Let us assume that $l > k$,

$$\langle v^k_{nm}, v^l_{n-2k+2l\, m}\rangle \neq 0 \tag{27}$$

and k is the smallest possible with that property. Now,

$$\langle A_\alpha . v^k_{nm}, v^{l-1}_{n-2k+2l\, m}\rangle = \langle -(k-1)v^{k-1}_{nm} + (n-k)v^{k+1}_{nm}, v^{l-1}_{n-2k+2l\, m}\rangle = 0.$$

By (26),

$$\langle v^k_{nm}, A^*_\alpha . v^{l-1}_{n-2k+2l\, m}\rangle = \langle v^k_{nm}, (l-2)v^{l-2}_{n-2k+2l\, m} + (n-2k+l+1)v^l_{n-2k+2l\, m}\rangle \neq 0.$$

It gives a contradiction to assumption (27). □

Let us assume that unitary $(\mathfrak{g}, K)$ module V is given. We want to find relationship among coefficients a_{nm}, b_{nm}, c_{nm} and d_{nm}. Relation

$$\langle A_{\alpha+\beta} . v^k_{n-1\, m-3}, v^k_{nm}\rangle = \langle v^k_{n-1\, m-3}, A^*_{\alpha+\beta} . v^k_{nm}\rangle,$$

produces

$$a_{n-1\, m-3}\|v^k_{nm}\|^2 = -\frac{n-k}{n-1}\overline{d_{nm}}\|v^k_{n-1\, m-3}\|^2. \tag{28}$$

Calculation is straightforward. Operator $A_{\alpha+\beta}$ is given by (7), (8) and (9), $A^*_{\alpha+\beta}$ by (26) and Lemma 15 is used for both sides. Relation (28), at first glance, does not look good. Namely, it gives a relationship between $a_{n-1\, m-3}$ and d_{nm}, but it depends on k. Let us write it for $k = 1$. It transforms to

$$a_{n-1\, m-3}\|v^1_{nm}\|^2 = -\overline{d_{nm}}\|v^1_{n-1\, m-3}\|^2. \tag{29}$$

One can apply (6) on both sides and obtain

$$a_{n-1\, m-3}\binom{n-1}{k-1}\|v^k_{nm}\|^2 = -\overline{d_{nm}}\binom{n-2}{k-1}\|v^k_{n-1\, m-3}\|^2$$

and it is easy to recognize (28). Hence, it is enough to consider (29). Relation

$$\langle A_{\alpha+\beta} . v^{k+1}_{n+1\, m-3}, v^k_{nm}\rangle = \langle v^{k+1}_{n+1\, m-3}, A^*_{\alpha+\beta} . v^k_{nm}\rangle,$$

produces

$$\frac{k}{n}c_{n-1\, m-3}\|v^k_{nm}\|^2 = -\overline{b_{nm}}\|v^{k+1}_{n+1\, m-3}\|^2.$$

Using (5) it transforms to

$$\frac{k}{n}c_{n+1\, m-3}\|v^k_{nm}\|^2 = -\overline{b_{nm}}\frac{k}{n+1-k}\|v^k_{n+1\, m-3}\|^2.$$

and

$$c_{n+1\,m-3}\|v^k_{nm}\|^2 = -\overline{b_{nm}}\frac{n}{n+1-k}\|v^k_{n+1\,m-3}\|^2.$$

Again, it is enough to consider this expression for $k=1$,

$$c_{n+1\,m-3}\|v^1_{nm}\|^2 = -\overline{b_{nm}}\|v^1_{n+1\,m-3}\|^2. \tag{30}$$

We continue and consider relations

$$\langle A_{\alpha+\beta}.v^{k-1}_{n-1\,m+3}, v^k_{nm}\rangle = \langle v^{k-1}_{n-1\,m+3}, A^*_{\alpha+\beta}.v^k_{nm}\rangle,$$

and

$$\langle A_{\alpha+\beta}.v^k_{n+1\,m+3}, v^k_{nm}\rangle = \langle v^k_{n+1\,m+3}, A^*_{\alpha+\beta}.v^k_{nm}\rangle.$$

However, they do not produce new relation among coefficients a_{nm}, b_{nm}, c_{nm} and d_{nm}. The same procedure can be repeated for $B_{\alpha+\beta}$, A_β and B_β but is will not produce new relation. Hence, only (29) and (30) have to be satisfied.

Now, let us go in opposite direction. We want to find conditions on coefficients a_{nm}, b_{nm}, c_{nm} and d_{nm} which will lead us to unitary (irreducible) $(\mathfrak{g}, K)$ module V. Lemma 15 shows that the inner product on V such that K acts by unitary operators is given by expressions $\|v^1_{nm}\|^2$. Relation (29), for irreducible V, shows that

$$a_{nm}d_{n+1\,m+3} \in (-\infty, 0) \tag{31}$$

and (30) shows that

$$b_{nm}c_{n+1\,m-3} \in (-\infty, 0). \tag{32}$$

One should compare (31) and (32) with (4). We claim that it is enough to satisfy (31) and (32). Hence, one has to define the "norm" for each V_{nm} such that (29) and (30) are satisfied. The expression "norm" of V_{nm}, by (6), means $\|v^1_{nm}\|^2$. It is enough to start with any K type V_{nm} and then define norms of other K types using (29) and (30). We have to prove that this construction is good. Let us assume that the norm of V_{nm} is given. We have to show that norms of $V_{n+2\,m}$, $V_{n\,m+6}$, $V_{n\,m-6}$ and $V_{n-2\,m}$ are well defined. The norm of $V_{n+2\,m}$ can be calculated in two different ways. Using (30) and (29), one obtains

$$\|v^1_{n+2\,m}\|^2 = -\frac{c_{n+2\,m}}{\overline{b_{n+1\,m+3}}}\|v^1_{n+1\,m+3}\|^2 = \frac{c_{n+2\,m}}{\overline{b_{n+1\,m+3}}}\cdot\frac{\overline{d_{n+1\,m+3}}}{a_{nm}}\|v^1_{nm}\|^2.$$

Similarly,

$$\|v^1_{n+2\,m}\|^2 = -\frac{\overline{d_{n+2\,m}}}{a_{n+1\,m-3}}\|v^1_{n+1\,m-3}\|^2 = \frac{\overline{d_{n+2\,m}}}{a_{n+1\,m-3}}\cdot\frac{c_{n+1\,m-3}}{\overline{b_{nm}}}\|v^1_{nm}\|^2.$$

It remains to show that

$$\frac{c_{n+2\,m}}{\overline{b_{n+1\,m+3}}} \cdot \frac{\overline{d_{n+1\,m+3}}}{a_{nm}} = \frac{\overline{d_{n+2\,m}}}{a_{n+1\,m-3}} \cdot \frac{c_{n+1\,m-3}}{\overline{b_{nm}}}.$$

It follows from (17) and (18). The norm of $V_{n\,m+6}$ can be also calculated in two different ways. Using (30) and (29), one obtains

$$\|v^1_{n\,m+6}\|^2 = -\frac{\overline{b_{n\,m+6}}}{c_{n+1\,m+3}}\|v^1_{n+1\,m+3}\|^2 = \frac{\overline{b_{n\,m+6}}}{c_{n+1\,m+3}} \cdot \frac{\overline{d_{n+1\,m+3}}}{a_{nm}}\|v^1_{nm}\|^2.$$

Similarly,

$$\|v^1_{n\,m+6}\|^2 = -\frac{\overline{d_{n\,m+6}}}{a_{n-1\,m+3}}\|v^1_{n-1\,m+3}\|^2 = \frac{\overline{d_{n\,m+6}}}{a_{n-1\,m+3}} \cdot \frac{\overline{b_{n-1\,m+3}}}{c_{nm}}\|v^1_{nm}\|^2.$$

It remains to show that

$$\frac{\overline{b_{n\,m+6}}}{c_{n+1\,m+3}} \cdot \frac{\overline{d_{n+1\,m+3}}}{a_{nm}} = \frac{\overline{d_{n\,m+6}}}{a_{n-1\,m+3}} \cdot \frac{\overline{b_{n-1\,m+3}}}{c_{nm}}.$$

It follows again from (17) and (18). Remaining two cases can be shown similarly.

It remains to notice that we have shown that

$$\langle C.w, v^1_{nm}\rangle = \langle w, (-C).v^1_{nm}\rangle$$

for $C = A_\beta$, B_β, $A_{\alpha+\beta}$ and $B_{\alpha+\beta}$ and $w = v^1_{n\pm1,m\pm3}$ if (31) and (32) are satisfied. By Lemma 15, it is enough since inner product is equal to 0 in all other cases. Hence operators A_β ,B_β, $A_{\alpha+\beta}$ and $B_{\alpha+\beta}$ are unitary. We have proved

Theorem 16. $(\mathfrak{g}, K)$ *module* V *is unitary if and only if* (31) *and* (32) *are satisfied.*

Remark 17. *One should compare statements of Theorem 16 and (4). Each time unitary action is obtained if certain products are real negative numbers. We hope that similar statement will be valid for some other real reductive groups.*

Now, we want to apply Theorem 16 on Theorem 10 and find all irreducible unitary $(\mathfrak{g}, K)$ modules. Firstly, we will analyze modules $V(c, 2t)$ and concentrate on the component which contains $V_{1\,2t}$. In the second step we will analyze modules $W(r, s)$ for $r > 1$. Theorem 16 says that expressions given by (22) and (23) have to be negative. Since $\dfrac{p+1}{p+q+2} > 0$ and $\dfrac{q+1}{p+q+2} > 0$ it reduces to

$$2c - (p+1)t - p(p+2) < 0 \tag{33}$$

and

$$2c + (q+1)t - q(q+2) < 0. \tag{34}$$

We want to find all values of c such that (33) and (34) are satisfied. If $t \geq 0$ then it is enough to consider only (34). If $t < 0$ then it is enough to consider only (33). The symmetry shows that it is enough to consider only one case. Hence, we will consider the case when $t \geq 0$. Since our expression is a quadratic polynomial in variable q we will consider situations when $t = 0$ and $t = 1$ separately (when the x coordinate of the vertex of the parabola is negative). Since $q \in \mathbb{N}$, we will consider situations $t = 2k$, $k \in \mathbb{N}$ and $t = 2k+1$, $k \in \mathbb{N}$ separately.

When $t = 0$, (34) reduces to $-q^2 - 2q + 2c < 0$ and it is fulfilled for $c < 0$. Let us define

$$c(0) = 0.$$

Hence $V(c, 0)$ is irreducible unitary for $c \in (-\infty, c(0))$. For $c = 0$, $V(c(0), 0)$ is reducible. Let us denote by $U(0)$ irreducible submodule which contains V_{10}. Then $U(0) = V_{10}$ is one-dimensional module.

When $t = 1$, (34) transforms to $-q^2 - q + 2c + 1 < 0$ and we set

$$c(1) = -\frac{1}{2}.$$

Hence, $V(c, 2)$ is irreducible unitary for $c \in (-\infty, c(1))$ and $V(c(1), 2)$ is reducible. Let $U(2)$ be an irreducible submodule which contains V_{12}. Then $U(2)$ contains K types of the form

$$\{V_{1+p\,2+3p} \,|\, p \in \mathbb{N} \cup \{0\}\}.$$

When $t = 2k$ for $k \in \mathbb{N}$, (34) shows that

$$c(2k) = -\frac{k^2+1}{2}$$

and $V(c, 4k)$ is irreducible unitary for $c \in (-\infty, c(2k))$. It is also possible that $V(c, 4k)$ has an irreducible unitary submodule which contains $V_{1\,4k}$. Relation (34) transforms to

$$2c + (q+1)t - q(q+2) = -(q-l)(q - (2(k-1) - l))$$

for $l \in \{0, \ldots, k-1\}$ and it produces

$$c(l, 2k) = \frac{l^2 - 2(k-1)l - 2k}{2}.$$

The submodule of $V(c(l,t), 2t)$ which contains $V_{1\,4k}$ for $l \in \{0, \ldots, k-1\}$ will be denoted by $U(l, 2t)$. It contains K types of the form

$$\{V_{1+p+q\,2t+3p-3q} \,|\, p \in \mathbb{N} \cup \{0\}, q \in \{0, \ldots, l\}\}. \tag{35}$$

When $t = 2k+1$ for $k \in \mathbb{N}$, (34) shows that

$$c(2k+1) = -\frac{k^2+k+1}{2}$$

and $V(c, 2(2k+1))$ is irreducible unitary for $c \in (-\infty, c(2k+1))$. Similarly as in a previous case, it is possible to find irreducible unitary submodules of $V(c, 2(2k+1))$ for some c. Again, (34) transforms to

$$2c + (q+1)t - q(q+2) = -(q-l)(q-(2k-1-l))$$

for $l \in \{0, \ldots, k-1\}$ and it produces

$$c(l, 2k+1) = \frac{l^2 - (2k-1)l - 2k - 1}{2}.$$

The submodule of $V(c(l,t), 2t)$ which contains $V_{1\,2(2k+1)}$ for $l \in \{0, \ldots, k-1\}$ will be denoted by $U(l, 2t)$ and it contains K types of the form (35).

The situation is very similar for $t < 0$. One has to use (33) instead of (34). It is easy to see that $c(t) = c(-t)$ and $c(l,t) = c(l,-t)$ for $l \in \{0, \ldots, \lfloor\frac{-t}{2}\rfloor - 1\}$. The set of K types of $U(-2)$ is $\{V_{1+q\,-2+3q} \,|\, q \in \mathbb{N} \cup \{0\}\}$ and the set of K types of $U(l, 2t)$ is

$$\{V_{1+p+q\,2t+3p-3q} \,|\, p \in \{0, \ldots, l\}, q \in \mathbb{N} \cup \{0\}\}$$

for $l \in \{0, \ldots, \lfloor\frac{-t}{2}\rfloor - 1\}$.

Now, let us consider modules $W(r,s)$ for $r > 1$. Theorem 10 claims that $W(r,s)$ is a submodule of some $V(c, 2t)$. Hence, (22) and (23) can be applied. Since $r > 1$, expressions given in (22) and (23) are equal to 0 in the previous step. Hence, it is enough to find when these two expressions are equal or less to 0.

We have mentioned that the system given by (12) and (13) in variables $a_{rs}d_{r+1\,s+3}$ and $b_{rs}c_{r+1\,s-3}$ ($c_{rs}b_{r-1\,s+3} = 0$ and $d_{rs}a_{r-1\,s-3} = 0$) has a unique solution

$$a_{rs}d_{r+1\,s+3} = -\frac{r(s+r+1)}{2(r+1)} \quad \text{and} \quad b_{rs}c_{r+1\,s-3} = \frac{r(s-r-1)}{2(r+1)}.$$

Since, $r > 0$ and $r+1 > 0$ it remains to consider $s+r+1 \geq 0$ and $s-r-1 \leq 0$. It can not happen that both expressions are equal to 0. If $s+r+1 = 0$ then $s-r-1 < 0$, $W(r,s)$ is reducible and the submodule which contains V_{rs} will be denoted by $Z(s)$ where $s \in -(\mathbb{N} \setminus \{1\})$. The set of K types of $Z(s)$ is

$$\{V_{-s-1+q\,s-3q} \,|\, q \in \mathbb{N} \cup \{0\}\}.$$

If $s-r-1=0$ then $s+r+1>0$, $W(r,s)$ is reducible and the submodule which contains V_{rs} will be denoted by $Z(s)$ where $s \in \mathbb{N} \setminus \{1\}$. The set of K types of $Z(s)$ is

$$\{V_{s-1+p\, s+3p} \,|\, p \in \mathbb{N} \cup \{0\}\}.$$

Finally, if $s+r+1>0$ and $s-r-1<0$ then $W(r,s)$ is unitary irreducible and the set K types is

$$\{V_{r+p+q\, s+p+q} \,|\, p, q \in \mathbb{N} \cup \{0\}\}.$$

Hence, we have proved

Theorem 18. *Irreducible unitary $(\mathfrak{g}, K)$ modules are parametrized as follows*

1. $V(c, 2t)$, $t \in \mathbb{Z}$, $c \in (-\infty, c(t))$,
2. $U(0)$, $U(2)$, $U(-2)$ *and* $U(l, 2t)$, $t \in \mathbb{Z} \setminus \{0, 1, -1\}$, $l \in \{0, \ldots, \left\lfloor \frac{|t|}{2} \right\rfloor - 1\}$,
3. $W(r,s)$, $s+r+1>0$ *and* $s-r-1<0$,
4. $Z(s)$, $s \in \mathbb{Z} \setminus \{-1, 0, 1\}$.

All these $(\mathfrak{g}, K)$ modules are nonequivalent.

One can compare Theorem 18 and results in [4]. Irreducible unitary representations of $SU(2,1)$ are given on the page 185 by the expression

$$W' = A \cup B'_+ \cup B'_- \cup C'_+ \cup C'_- \cup D' \cup E_+ \cup E_- \cup F.$$

Here $A = \{t \in \mathbb{R} \,|\, t > 0\}$ (defined on the page 182 for the universal covering group $\widetilde{SU(2,1)}$ of $SU(2,1)$) stands for irreducible unitary representations with the spectrum $\Gamma_0 = \{(\frac{r}{2} + \frac{s}{2}, r-s) \,|\, r, s \in \mathbb{N} \cup \{0\}\}$, defined in the Proposition 1, case 8 and discussed again on the page 182. The pair $(\frac{r}{2} + \frac{s}{2}, r-s)$ denotes the K type of dimension $r+s+1$. We have the similar notation. The set A corresponds to the set $\{V(c,0) \,|\, c < c(0) = 0\}$ which is mentioned in the first part of the Theorem 18. We can continue with B'_+, but it is clear that the correspondence is not simple.

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