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On classification of morphisms by the box-homotopy

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Abstract

In [1] the authors proposed a generalization of the notion of homotopy, a relation called to be box-homotopic, proven to be an equivalence relation on $Top(X, Y)$ and well-adjusted with the composition. In this article we prove that all the mappings of $Top(X, Y)$ are box-homotopic, that is, the classification of morphisms by the box-homotopy relation is the coarsest.

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1 Introduction

We refer to the article [1] in which we observed the possibility to reduce the category pro^*-$HTop$ of inverse systems of topological spaces and corresponding morphisms consisting of sequences of homotopy classes of mappings between terms in inverse systems. The idea for the reduction was to map inverse systems of topological spaces as objects of pro^*-category to inverse systems of so called reduced powers of topological spaces as objects of the *pro*-category. A reduced power of a topological space X is the product of the space X by itself countably many times, endowed the box topology, and reduced to a quotient space by an equivalence relation saying that two sequences of elements of X are related if they differ only on finite number of coordinates. Thus, we needed to represent the morphisms of pro^*-category as morphisms of *pro*-category between inverse systems of reduced powers. The construction of the reduction for the category *pro**-*Top* is straightforward, but the problem arose with the category *pro**-*HTop* which is of our main interest since its subcategory *pro**-*HPol* is the realization category for the topological coarse shape category. The problem is that in the case of countable products of topological spaces, when two sequences of maps are homotopic on every coordinate of the product, its products don't need to be homotopic, and consequently its induced morphisms on reduced powers don't need to be homotopic. That was the reason for proposing box-homotopy, a generalization of the notion of homotopy. In [1] we proved that box-homotopy is an equivalence relation on $Top(X, Y)$ and that it is well-adjusted with the composition, enabling us to define a quotient category of the category Top by box-homotopy relation on morphisms. Here we show that the classification of morphisms by box-homotopy relation yields the singleton quotient sets.

2 Preliminaries

A reduced power $(\widetilde{X}, \widetilde{\mathcal{T}})$ of a topological space $(X, \mathcal{T})$ is the product of the space $(X, \mathcal{T})$ by itself countably many times, given the box topology, and reduced to a quotient space by an equivalence relation saying that two sequences of elements in X are related if they differ only on finite number of coordinates.

The box topology on the cartesian product $\prod\limits_{\mathbb{N}} X$ proved to be a better choice than the commonly used product topology. In fact, if $\prod\limits_{\mathbb{N}} X$ has the product topology $\mathcal{T}_p$, then the quotient topology on $\widetilde{X}$ is indiscrete, as we can see in the following.

Recall that a collection

$$\mathcal{B}_p = \left\{ \prod_{\mathbb{N}} U_n \mid U_n \in \mathcal{T}, \, U_n \neq X \text{ for only finitely many } n \right\}$$

is a basis for $\mathcal{T}_p$. If $p\colon \prod\limits_{\mathbb{N}} X \to \widetilde{X}$ is a projection,

$$p((x_n)) = [(x_n)] = \{(x'_n) \mid x_n \neq x'_n \text{ for only finitely many } n \},$$

and $V = \prod\limits_{\mathbb{N}} U_n \in \mathcal{B}_p$, then

$$\begin{aligned} p(V) = p\left(\prod_{\mathbb{N}} U_n\right) &= \left\{ [(x_n)] \in \widetilde{X} \mid x_n \notin U_n \text{ for only finitely many } n \right\} \\ &= \left\{ [(x_n)] \in \widetilde{X} \mid x_n \notin X \text{ for only finitely many } n \right\} = \widetilde{X}. \end{aligned}$$

If $U \subseteq \widetilde{X}$ is open in $\widetilde{X}$, $U \neq \emptyset$, then $p^{-1}(U)$ is open in $\prod_{\mathbb{N}} X$. Therefore, $p^{-1}(U) = \bigcup_{V \in \mathcal{S}} V$, $\mathcal{S} \subseteq \mathcal{B}_p$. We have

$$p\left(p^{-1}(U)\right) = p\Big(\bigcup_{V \in \mathcal{S}} V\Big) = \bigcup_{V \in \mathcal{S}} p(V) = \widetilde{X}.$$

Since $p\left(p^{-1}(U)\right) \subseteq U$, we get $\widetilde{X} \subseteq U$, and consequently $\widetilde{X} = U$, which confirmes that the quotient topology on $\widetilde{X}$ is indiscrete.

As can be seen in [1], if the product $\prod_{\mathbb{N}} X$ is endowed with the box-topology, the coresponding quotient space $\big(\widetilde{X}, \widetilde{\mathcal{T}}\big)$ has the topology $\widetilde{\mathcal{T}}$ generated by the basis

$$\widetilde{\mathcal{B}} = \Big\{ \nabla_{\mathbb{N}} U_n \mid U_n \in \mathcal{T} \text{ for every } n \in \mathbb{N} \Big\},$$

where $\nabla_{\mathbb{N}} U_n \equiv p\big(\prod_{\mathbb{N}} U_n\big) = \Big\{ [(x_n)] \in \widetilde{X} \mid x_n \notin U_n \text{ for only finitely many } n \Big\}$ for $U_n \subseteq X$, $n \in \mathbb{N}$.

The space $\big(\widetilde{X}, \widetilde{\mathcal{T}}\big)$, or abbreviated $\widetilde{X}$, can be found in studies of Rudin and Kunen (for example in [3] and [2]), and is reffered to as a reduced box product or a reduced power.

Remark 1. *When a product space $\prod_{\mathbb{N}} X$ is endowed with the box topology, it is usually denoted by $\square_{\mathbb{N}} X$, and when it is endowed with the product topology, it is usually denoted by X^{ω}.*

Let us list some of the properties of $\widetilde{X}$.

Theorem 2. *The intersection of a countable collection of open sets in $\widetilde{X}$ is an open set in $\widetilde{X}$.*

Proof. The proof is given in [1]. □

Remark 3. *In literature, when a space has a property that the intersection of a countable collection of its open subsets is open, it is called a P-space in the sense of Gillman-Henriksen.*

Therefore, $\widetilde{X}$ is a P-space, and yet, generally it is not discrete. For example, a reduced power of Sierpiński space $X = \{0, 1\}$ with a topology $\mathcal{T} = \{\emptyset, \{1\}, \{0, 1\}\}$ is not discrete because $\{[(x_n)]\}$ is not open in $\widetilde{X}$ whenever $x_n \neq 1$ for infinitely many n.

Theorem 4. *If X is a T_i space for $i = 0, 1, 2$ or 3, then $\widetilde{X}$ is also a T_i space.*

Proof. See [1]. □

Theorem 5. *In $\widetilde{X}$, regularity and complete regularity are equivalent.*

Proof. Recall that if $\widetilde{X}$ is a regular space, then for every $x \in \widetilde{X}$ and $F \subseteq \widetilde{X}$ closed in $\widetilde{X}$ not containing x we can construct a sequence (V_n) of open sets in $\widetilde{X}$ such that

$$x \in V_n \subseteq \widetilde{X} \backslash F \ , \quad \text{for every } n \in \mathbb{N}$$

and

$$\overline{V}_{n+1} \subseteq V_n \ , \quad \text{for every } n \in \mathbb{N}.$$

Then, $x \in \bigcap\limits_{n \in \mathbb{N}} V_n$. The space $\widetilde{X}$ is a P-space, implying that the set $\bigcap\limits_{n \in \mathbb{N}} V_n$ is open in $\widetilde{X}$. It is also closed since $\bigcap\limits_{n \in \mathbb{N}} V_n = \bigcap\limits_{n \in \mathbb{N}} \overline{V}_n$, and it is disjoint with F.

Consequently, the map $f \colon \widetilde{X} \to [0,1]$ defined by

$$f(x) = \begin{cases} 1\,, & x \in \bigcap\limits_{n \in \mathbb{N}} V_n, \\ 0\,, & x \in \widetilde{X} \backslash \bigcap\limits_{n \in \mathbb{N}} V_n, \end{cases}$$

is continuous. Therefore, $\widetilde{X}$ is completely regular. □

Theorem 6. *If $\widetilde{X}$ is a regular space, it is totally disconnected.*

Proof. Let $\widetilde{X}$ be a regular space. Then, as seen in the proof of the Theorem 2, for every $x \in \widetilde{X}$ and $U \subseteq \widetilde{X}$ open in $\widetilde{X}$ such that $x \in U$ there is a sequence (V_n) of open sets in $\widetilde{X}$ such that $x \in \bigcap\limits_{n \in \mathbb{N}} V_n \subset U$. The set $\bigcap\limits_{n \in \mathbb{N}} V_n$ is open and closed in $\widetilde{X}$, and such sets (for various $x \in \widetilde{X}$) form a clopen base of the topology of $\widetilde{X}$.

Consequently, since every point in such a space can be separated from every other point by a clopen neighbourhood, $\widetilde{X}$ is totally disconnected. □

In [1] we have assigned to a sequence of morphisms (f^n) in $Top(X,Y)$ a mapping denoted by $\nabla\limits_{\mathbb{N}} f^n$ from $\widetilde{X}$ to $\widetilde{Y}$, induced by $\prod\limits_{\mathbb{N}} f^n \colon \square_{\mathbb{N}} X \to \square_{\mathbb{N}} Y$, $\prod\limits_{\mathbb{N}} f^n((x_n)) = (f^n(x_n))$, for $(x_n) \in \square_{\mathbb{N}} X$. It proves out that it is well defined and contuinous, so it is a morphism in $Top(\widetilde{X}, \widetilde{Y})$.

We also wanted to assign to a sequence of morphisms $([f^n])$ in $HTop(X,Y)$ an analogous association from $\widetilde{X}$ to $\widetilde{Y}$. The obvious choice would be the homotopy class of $\underset{\mathbb{N}}{\nabla} f^n$. There we have encountered a problem because if $(f^n), (f'^n) \colon X \to Y$ are two sequences of continuous maps such that f^n is homotopic to f'^n for every $n \in \mathbb{N}$, then $\underset{\mathbb{N}}{\nabla} f^n$ doesn't need to be homotopic to $\underset{\mathbb{N}}{\nabla} f'^n$. The following example confirms that.

Example 7. *Let $(f^n), (g^n) \colon I \to I$, where $I = [0,1]$, such that $f^n = 0$ and $g^n = 1$ for every $n \in \mathbb{N}$. The maps f^n and g^n are homotopic for every $n \in \mathbb{N}$. But, $\underset{\mathbb{N}}{\nabla} f^n$, $\underset{\mathbb{N}}{\nabla} g^n \colon \widetilde{I} \to \widetilde{I}$ are not homotopic. Indeed, if $H \colon \widetilde{I} \times I \to \widetilde{I}$ is a homotopy between $\underset{\mathbb{N}}{\nabla} f^n$ and $\underset{\mathbb{N}}{\nabla} g^n$, then $H([(x_n)], 0) = [(0, 0, \dots)]$, and $H([(x_n)], 1) = [(1, 1, \dots)]$ for every $[(x_n)] \in \widetilde{I}$. Then for every $[(x_n)]$, a map $I \to \widetilde{I}$ defined by $t \mapsto H([(x_n)], t)$ is a path from $[(0, 0, \dots)]$ to $[(1, 1, \dots)]$ in $\widetilde{I}$, and that is a contradiction because $\widetilde{I}$ is totally disconnected according to previous since I is a regular space.*

To overcome the problem, we proposed a generalisation of the notion of homotopy, a relation that we called box-homotopy. The idea was to assign to a sequence of morphisms $([f^n])$ in $HTop(X, Y)$ a box-homotopy class of $\underset{\mathbb{N}}{\nabla} f^n$.

3 On box-homotopy

Definition 8. *Let f, g be morphisms in $Top(X, Y)$. We will say that f and g are box-homotopic, $f \underset{\square}{\sim} g$, if there is a continuous mapping $H \colon X \times \underset{\mathbb{N}}{\square} I \to Y$ satisfying*

$$H(x, 0, 0, 0, \dots) = f(x),$$
$$H(x, 1, 1, 1, \dots) = g(x),$$

for every $x \in X$.

As shown in [1], all homotopic maps are box-homotopic, and the converse does not hold. Also, it proves out that box-homotopy is an equivalence relation on $Top(X, Y)$, and that it is well adjusted with the composition, enabling us to define a quotient category of the category Top by box-homotopy relation on morphisms.

Theorem 9. *For all $f, g \in Top(X, Y)$, it holds $f \underset{\square}{\sim} g$. Consequently, the classification of morphisms by the box-homotopy relation yields the singleton quotient sets.*

Proof. Let $f, g \in Top(X, Y)$. We are to show that $f \underset{\square}{\sim} g$.

Let

$$A = \left\{ (x_n) \in \underset{\mathbb{N}}{\square} I \mid (x_n) \text{ converges to zero } \right\} \subset \underset{\mathbb{N}}{\square} I \quad \text{and}$$
$$B = \left\{ (x_n) \in \underset{\mathbb{N}}{\square} I \mid (x_n) \text{ doesn't converge to zero } \right\} \subset \underset{\mathbb{N}}{\square} I.$$

The set A is open since every $(x_n) \in A$ has an open neighbourhood

$$\underset{\mathbb{N}}{\square} \left(\left\langle x_n - \frac{1}{2^n}, x_n + \frac{1}{2^n} \right\rangle \cap I \right) \subset A.$$

B is also open since every $(x_n) \in B$ has an open neighbourhood

$$\underset{\mathbb{N}}{\square} \left(\left\langle x_n - \frac{1}{2^n}, x_n + \frac{1}{2^n} \right\rangle \cap I \right) \subset B.$$

The sets A and B are mutually disjoint and $A \cup B = \underset{\mathbb{N}}{\square} I$. Therefore, $\{A, B\}$ is a separation of $\underset{\mathbb{N}}{\square} I$.

Now we can define a mapping $H\colon X \times \underset{\mathbb{N}}{\square} I \to Y$ by

$$H|_{X \times A} = f \circ p_1,$$
$$H|_{X \times B} = g \circ p_1,$$

where $p_1\colon X \times \underset{\mathbb{N}}{\square} I \to X$ is the projection to the first coordinate. It is continuous according to the pasting lemma, and it is a box-homotopy from f to g. Therefore, $f \underset{\square}{\simeq} g$. $\square$

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