ACTA MATHEMATICA SPALATENSIA
Vol.1 (2021) 87–96
https://doi.org/10.32817/ams.1.1.7

Received: 06 Feb 2020
Accepted: 24 Feb 2020

Complete forcing numbers of rectangular polynominoes

Hong Chang, Yong-De Feng, Hong Bian, Shou-Jun Xu

Abstract

Let G be a graph with edge set $E(G)$ that admits a perfect matching M. A *forcing set* of M is a subset of M contained in no other perfect matchings of G. A *complete forcing set* of G, recently introduced by Xu et al. [Complete forcing numbers of catacondensed hexagonal systems, *J. Combin. Optim.* 29(4) (2015) 803-814], is a subset of $E(G)$ on which the restriction of any perfect matching M is a forcing set of M. The minimum possible cardinality of complete forcing sets of G is the *complete forcing number* of G. In this article, we discuss the complete forcing number of rectangular polyominoes (or grids), i.e., the Cartesian product of two paths of various lengths, and show explicit formulae for the complete forcing numbers of rectangular polyominoes in terms of the lengths.

Keywords: perfect matching, forcing set, complete forcing set, complete forcing number, polyomino
2010 Math. Subj. Class.: 05C90

1 Introduction

The concept of forcing set and forcing number of kekulé structures (i.e., perfect matchings) have arisen in the study of resonance theory in the chemical literature [8, 10] and subsequently in purely graph-theoretical literature [7]. We refer the interested reader to a recent survey [4] and the references cited therein.

Forcing sets and forcing numbers of Kelulé structures of a graph G with edge set $E(G)$ are defined by the "local" approach, i.e., defined with respect to a particular Kelulé structure of G. On the other hand, Klein and Randić [8] proposed *the degree of freedom* of a graph from the "global" point of view, defined as the sum of forcing numbers over all Kelulé structures of a graph, and showed by evidence that the degree of freedom of a chemical graph actually measures chemical characteristics of the corresponding modular graphs distinct from those measured by a couple of common resonance-energy estimators. Combining the "forcing" and "global" ideas, Xu et al. [13] first proposed the concepts of complete forcing sets and complete forcing number of a graph with respect to Kelulé structures from the "global" point of view (see the next section for the definitions). To some extent, the complete forcing number of a graph gives some sort of identification of the minimal amount of "information" required to specify forcing sets of kekulé structures of the graph. Recently Xu et al. gave some initial results about complete forcing sets of a graph, some calculations and algorithms for complete forcing numbers of some typical graphs [3, 11, 12, 13].

Since the enumeration problem for perfect matchings in general graphs (even in bipartite graphs) is NP-hard [9], the computation of complete forcing numbers is very difficult. It is interesting to relate the complete forcing number of a graph G to some structural properties of G. An alternative approach is to seek for the ways to express the complete forcing numbers of composite graphs in terms of their components. In [13], Xu et al. considered the composite graph G of two catacondensed benzenoid systems G_1 and G_2 by identifying an edge with degree-two endpoints in G_1 with an edge of the same type in G_2 and express the complete forcing number of G in terms of the complete forcing numbers of G_1 and G_2. Recently, Chan and Xu [3] generalized to the composite graph of the same type of two general graphs and as its application, gave a linear-time algorithm for computing the complete forcing numbers of catacondensed benzenoid systems.

Another important class of composite graphs are Cartesian products $G_1 \Box G_2$ of two graphs G_1 and G_2. For general graphs G_1 and G_2 there seem to be no simple formulae for the complete forcing number of $G_1 \Box G_2$, but if both G_1 and G_2 are paths of various lengths, and one of paths has an even number of vertices, then the complete forcing number of their Cartesian product $G_1 \Box G_2$ can be explicitly expressed in terms of their lengths, which is the main result of this paper. In fact, such $G_1 \Box G_2$ are a special class of polyominoes (called *rectangular polyominoes*). If both paths have odd number of vertices, then their Cartesian product, namely, the corresponding grid graph does not admit perfect matching, which is not the case under consideration. At the present

time polyominoes are widely known by mathematicians, physicists and chemists and have been considered in many different applications [1].

The present paper is organized as follows. In the next section, we formally define complete forcing sets, the complete forcing number of a graph, along with other graph-theoretic terms relevant to our subject and some lemmas which will be used. In Section 3, we give an important theorem: Pick's theorem, and then give and prove our main result: explicit formulae for complete forcing numbers of rectangular polyominoes. We conclude this article in Section 4, and some problems of further research are also put forward.

2 Preliminaries

Each graph G with edge set $E(G)$ and vertex set $V(G)$ in this paper is simple and connected and has perfect matchings. For all terms and notations used but not defined here we refer the reader to the textbook [5].

A *polyomino* (*graph*) [2] is a connected finite subgraph of the infinite plane grid such that each interior face is surrounded by a regular square of unit size (called a *cell*) and each edge belongs to at least one cell. A polyomino is called *rectangular* if its outer boundary is exactly a rectangle with length $m-1$ and width $n-1$ for two positive integers m, n ($\geqslant 2$), denoted by $R_{m,n}$. Let G_1 and G_2 be two simple graphs. Their cartesian product $G_1 \Box G_2$ is the graph with vertex set $V(G_1) \times V(G_2) = \{(u_i, v_j) : u_i \in V(G_1), v_j \in V(G_2)\}$ in which two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$, or $v_1 = v_2$ and $u_1 u_2 \in E(G_1)$. An alternative definition of $R_{m,n}$ is the Cartesian product of two paths P_m and P_n, where P_i is a path with i vertices (see Fig. 1 for an example with $m = 7$ and $n = 6$). Denote the vertices in P_m and P_n by $\{1, 2, ..., m\}$ and $\{1, 2, ..., n\}$, respectively. For convenience, $R_{m,n}$ can be embedded in the plane by interpreting vertex label as a point in cartesian coordinate system, and the vertices of $R_{m,n}$ automatically have notation (i, j). Hence the lower left corner (i.e., the lower left vertex) has the coordinates $(1, 1)$, and the upper right corner has the coordinates (m, n). Note that $R_{m,n}$ is bipartite. In this paper, it is necessary to take that at least one of m and n is even, otherwise $R_{n,m}$ has no any perfect matching (see below for the definition), so we always assume that mn is even.

A *perfect matching* (or *kekulé structure*) of a graph is a set of disjoint edges that covers all vertices of the graph. An edge of G is termed *allowed* if it lies in some perfect matching of G and *forbidden* otherwise. A graph G is said to be *elementary* if all its allowed edges form a connected

subgraph of G. Note that if a connected bipartite graph is elementary, then every edge is allowed.

Let S_1, S_2 be two subsets of a set. The *symmetric difference*, denoted by $S_1 \oplus S_2$, of S_1 and S_2 is the set of elements belonging to exactly one of S_1 and S_2.

Let G be a connected graph with a perfect matching. A subgraph H of G is *nice* if $G - V(H)$ contains a perfect matching. It is obvious that an even cycle C of G is *nice* if and only if C is exactly the symmetric difference of some two perfect matchings M_1 and M_2 of G, i.e., $C = M_1 \oplus M_2$; we call $M_1 \cap C$ and $M_2 \cap C$ two *type-edges* of C. Alternatively, each type-edges of a nice cycle C is a perfect matching of C.

For rectangular polyominoes, there is a basic property as follows.

Theorem 1. *Let $R_{m,n}$ be a rectangular polyomino with mn even. Then $R_{m,n}$ has a perfect matching and is elementary.*

Proof. By the symmetry of m and n, we assume without loss of generality that n is even. We define a subset M of edge set of $R_{m,n}$: $\{(i, 2j-1)(i, 2j) \colon i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, \frac{n}{2}\}$. It is obvious that M is a perfect matching of $R_{m,n}$. Now we show that $R_{m,n}$ is elementary. It is straightforward in the case $m = 2$ or $n = 2$. We assume that $m, n \geqslant 3$ in the following.

First, we construct two new perfect matchings M_1 and M_2 in $R_{m,n}$. Let C be the boundary of $R_{m,n}$. If m equals to 3, then $R_{m,n} \setminus C$ is exactly a path of length $n-2$ with the unique perfect matching, denoted by M'. Otherwise $R_{m,n} \setminus C$ is exactly $R_{m-2,n-2}$. We denote by M' the perfect matching of $R_{m-2,n-2}$ constructed similar to M in $R_{m,n}$ (indicated by dotted lines in Fig. 1 It is obvious that C is nice. We denote by C_1, C_2 two type-edges of C, respectively, and then by $M_1 = C_1 \cup M'$, $M_2 =$

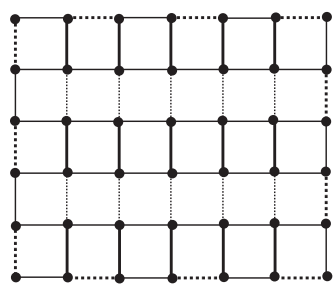

Fig. 1: Illustration for the proof of Theorem 1 with rectangular polyomino $R_{7,6}$. M' is indicated by dotted lines, part of M is indicated by bold lines, a group of alternative edges in C is indicated by bold dotted lines (C_1 or C_2 in the proof).

$C_2 \cup M'$, respectively. It is obvious that the set of edges $M \cup M_1 \cup M_2$ is the subset of all allowed edges in $R_{m,n}$. Since $M \cup M_1 \cup M_2$ consists of all edges in $R_{m,n}$ except all horizontal edges other than on C of $R_{m,n}$, it induces a connected subgraph of $R_{m,n}$ such that adding other allowed edges to it, which can not make it disconnected any more. Hence $R_{m,n}$ is elementary. □

Lemma 2. *[14] Let G be a plane bipartite graph with more than two vertices. Then the boundary of each face of G is nice if and only if G is elementary.*

Combined Theorem 1 with Lemma 2, we get

Theorem 3. *Let $R_{m,n}$ be a rectangular polyomino with mn even. Then each cell boundary in $R_{m,n}$ is nice.*

Let G be a connected graph with edge set $E(G)$ and a perfect matching M. A *forcing set* of M is a subset of M contained in no other perfect matchings of G. It follows that the empty set is a forcing set of M if and only if M is the unique perfect matching of G. A *complete forcing set* of G is a subset S of $E(G)$ such that restriction of M to S is a forcing set of M. Obviously, any set containing a complete forcing set of G, particularly $E(G)$, is also a complete forcing set of G. A complete forcing set of the smallest cardinality is called a *minimum complete forcing set*, and its cardinality is the *complete forcing number* of G, denoted by $\gamma(G)$.

As an illustrative example we consider K_4 shown in Fig. 2. It contains three different perfect matchings: $M_1 = \{e_1, e_4\}$, $M_2 = \{e_2, e_5\}$, $M_3 = \{e_3, e_6\}$. It is easy to see that the restriction of every perfect matching M on $S = \{e_1, e_2, e_3\}$ is a forcing set of M. Hence S is a complete forcing set of K_4. Since the intersection of S and every perfect matching is nonempty and $\{M_1, M_2, M_3\}$ is a partition of the edge set of K_4, S of cardinality 3 is a minimum complete forcing set. Hence, $\gamma(K_4) = 3$.

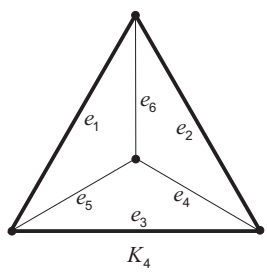

Fig. 2: A minimum complete forcing set $\{e_1, e_2, e_3\}$ in K_4 is indicated by bold lines. Hence $\gamma(K_4) = 3$.

We give a sufficient and necessary condition for a complete forcing set of a connected graph with a perfect matching.

Theorem 4. *[13] Let G be a connected graph with edge set $E(G)$ and a perfect matching. A set $S \subseteq E(G)$ is a complete forcing set of G if and only if, for any nice cycle C in G, the intersection of S and each type-edges of C is non-empty.*

3 Complete Forcing Number of $R_{m,n}$

In this section we give and prove the main result: explicit formulae for the complete forcing number of $R_{m,n}$ (i.e., Theorem 8). First, we give an important theorem (called Pick's theorem) as follows.

Theorem 5 (Pick's theorem)**.** *[6] Let P be a simple polygon constructed on a polyomino such that all the polygon's vertices are polyomino's vertices, let the number of polyomino's vertices in the interior of P be i, let the number of polyomino's vertices on the boundary of P be b. Then the area of P is given by*

$$A = \frac{b}{2} + i - 1.$$

Lemma 6. *Let $R_{m,n}$ be a rectangular polyomino with mn even. Then $\gamma(R_{m,n}) \leq \lfloor \frac{n}{2} \rfloor (m-1) + \lfloor \frac{m}{2} \rfloor (n-1)$.*

Proof. It is sufficient to give a complete forcing set S of cardinality $\lfloor \frac{n}{2} \rfloor (m-1) + \lfloor \frac{m}{2} \rfloor (n-1)$. First, we give such a set S and then show that S a complete forcing set.

We define S as (see Fig. 3)

$$\{(2i,j)(2i,j+1) \colon i = 1,2,\ldots,\left\lfloor \frac{m}{2} \right\rfloor, j = 1,2,\ldots,n-1\}$$
$$\text{(set of vertical edges)}$$
$$\cup \{(i,2j)(i+1,2j) \colon i = 1,2,\ldots,m-1, j = 1,2,\ldots,\left\lfloor \frac{n}{2} \right\rfloor\}.$$
$$\text{(set of horizontal edges)}$$

By a direct calculation one can verify that the cardinality of S is $\lfloor \frac{n}{2} \rfloor (m-1) + \lfloor \frac{m}{2} \rfloor (n-1)$.

In what follows we prove S is a complete forcing set of $R_{m,n}$. By Theorem 4, it is sufficient to prove S has a nonempty intersection with both of type-edges of each nice cycle C in $R_{m,n}$.

Case 1 C is exactly some regular square in $R_{m,n}$.

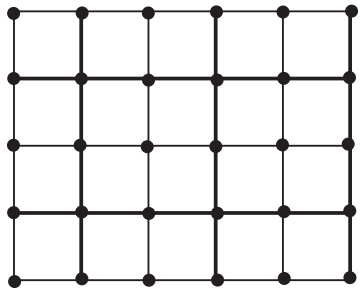

Fig. 3: Illustration for construction of a subset S of edges on $R_{6,5}$ in the proof of Lemma 6. S is indicated by bold lines.

By the construction of S, we can easily obtain that S shares exactly two adjacent edges with C, which, hence, belong to different type-edges of C. So S has a nonempty intersection with both type-edges of C.

Case 2 C is not a cell boundary cycle.

Claim $S \cap E(C) \neq \emptyset$.

We assume to the contrary that $S \cap E(C) = \emptyset$. Let G be the graph obtained from $R_{m,n}$ by deleting edges in S, then deleting all pendant edges (i.e., edges with an endpoint of degree one) and finally deleting all isolated vertices. Note that G has the similar structure to rectangular polyominoes, but the only difference is that squares in the former have side length 2, while ones in the latter has side length 1. Then C is contained completely in G by the assumption. We suppose that C encloses some region R in the plane, let i be the number of vertices of $R_{m,n}$ in the interior of C, b the number of vertices of $R_{m,n}$ on C, and let A be the area of R. Then A is divisible by four. Since the reduction of G by half produces a biparite polynomino G' and C in G is corresponding to a cycle G' of even length, b is divisible by four. By Theorem 5, i is odd. So $R_{m,n} - V(C)$ has no perfect matching, which contradicts that C is a nice cycle.

Let $e = uv$ be an edge in $S \cap E(C)$. By the construction of S, at least one vertex u or v, say u, there are two edges incident to u containing in S. So C shares two edges incident to u with S, which belong to two type-edges of C. Hence S has a nonempty intersection with both of type-edges of C. $\square$

Lemma 7. *Let $R_{m,n}$ be a rectangular polyomino with mn even. Then $\gamma(R_{m,n}) \geq \lfloor \frac{n}{2} \rfloor (m-1) + \lfloor \frac{m}{2} \rfloor (n-1)$.*

Proof. Let S be any complete forcing set of $R_{m,n}$. It is sufficient to show that the cardinality $|S|$ of S is at least $\lfloor \frac{n}{2} \rfloor (m-1) + \lfloor \frac{m}{2} \rfloor (n-1)$.

By Theorems 3 and 4, S has a non-empty intersection with each type-edges of each square h in $R_{m,n}$. Hence S contains at least one vertical (resp. horizontal) edge in h. Since there are $(n-1)$ (resp. $(m-1)$) squares in each row (resp. column) of squares in $R_{m,n}$, S contains at least $\lceil \frac{n-1}{2} \rceil = \lfloor \frac{n}{2} \rfloor$ (resp. $\lceil \frac{m-1}{2} \rceil = \lfloor \frac{m}{2} \rfloor$) vertical (resp. horizontal) edges for each row (resp. column) of squares. Note that n or m is possibly odd. As there are $m-1$ rows (resp. $n-1$ columns) of squares, S contains at least $\lfloor \frac{n}{2} \rfloor (m-1) + \lfloor \frac{m}{2} \rfloor (n-1)$ edges. □

Combined Lemmas 6 and 7, we can obtain our main result as follows.

Theorem 8. *Let $R_{m,n}$ be a rectangular polyomino with mn even. Then the complete forcing number of $R_{m,n}$ is $\lfloor \frac{n}{2} \rfloor (m-1) + \lfloor \frac{m}{2} \rfloor (n-1)$, i.e., $\gamma(R_{m,n}) = \lfloor \frac{n}{2} \rfloor (m-1) + \lfloor \frac{m}{2} \rfloor (n-1)$.*

4 Conclusions

In this paper, we consider complete forcing numbers of rectangular polyominoes and give their explicit formulae in terms of their structural parameters. Considering Cartesian product of two paths P_n and P_m as an alternative definition of rectangular polyominoes, the most natural next step would be to extend the results of Theorem 8 to cylinders and tori: i.e., to Cartesian product of $P_n \times C_m$ and $C_n \times C_m$ for nm even, where C_n is the cycle of length n. One can expect results expressing the complete forcing numbers of those graphs in terms of their sizes.

Acknowledgements. The authors would like to thank the editor and anonymous reviewers for their helpful comments and suggestions which lead to a considerably improved presentation. This work is partially supported by National Natural Science Foundation of China (Grant No. 12071194, 11761070, 61662079).

References

[1] L. Alonso, R. Cerf, *The three dimensional polyominoes of minimal area*, Electron. J. Comb. **3** (1996) #27.

[2] S. W. Golomb, *Polyominoes*, Charles Scribner's Sons, New York, 1965.

[3] W. H. Chan, S.-J. Xu, G. Nong, *A linear algorithm for computing complete forcing numbers of catacondensed hexagonal systems*, MATCH-Commun. Math. Comp. Chem. 74(1) (2015) 201–216.

[4] Z. Che, Z. Chen, *Forcing on perfect matchings-A survey*, MATCH Commun. Math. Comput. Chem. **66** (2011) 93–136.

[5] R. Diestel, *Graph theory*, Second Edition, Springer-Verlag, New York, 2000.

[6] B. Grunbaum, G. C. Shephard, *Pick's theorem*, Amer. Math. Monthly **100** (1993) 150–161.

[7] F. Harary, D. J. Klein and T. P. Živković, *Graphical properties of polyhexes: perfect matching vector and forcing*, J. Math. Chem. **6** (1991) 295–306.

[8] D.J. Klein and M. Randić, *Innate degree of freedom of a graph*, J. Comput. Chem. **8** (1987) 516–521.

[9] L. Lovász and M. Plummer, *Matching Theory*, in: Annals of Discrete Math., Vol. 29, North-Holland, Amsterdam, 1986.

[10] M. Randić, D. J. Klein, *Kekulé valence structures revisited. Innate degrees of freedom of π-electron couplings*, in: N. Trinajstić (Ed.), Mathematical and computational concepts in chemistry, Wiley, New York, 1985, 274–282.

[11] J.-M. Xiao, H. Bian, Y.-D. Feng, S.-J. Xu, *Complete forcing numbers of triangular benzenoid systems*, completed.

[12] S.-J. Xu, X.-S. Liu, W.H. Chan and H. Zhang, *Complete forcing numbers of primitive coronoids*, J. Combin. Optim. 32(1) (2016) 318-330.

[13] S.-J. Xu, H. Zhang, J. Cai, *Complete forcing numbers of catacondensed benzenoid*, J. Combin. Optim. 29(4) (2015) 803–814.

[14] H. Zhang and F. Zhang, *Plane elementary bipartite graphs*, Discrete Appl. Math. **105** (2000) 291–311.

Hong Chang
(1) School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou, Gansu 730000, China
(2) School of Mathematical Science & Institute of Mathematics, Nanjing Normal University, Nanjing, Jiangsu 210023, P. R. China

Yong-De Feng
(1) School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou, Gansu 730000, China
(2)College of Mathematics and Systems Science, Xinjiang University, Urumqi, Xinjiang 830046, P.R. China

Hong Bian
School of Mathematical Sciences, Xinjiang Normal University, Urumqi, Xinjiang 830054, P.R. China

Shou-Jun Xu
School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou, Gansu 730000, China
E-mail address: shjxu@lzu.edu.cn