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Some coarse properties of global morphisms of countable approximate groups

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Abstract

The notion of coarsely n-to-1 maps between metric spaces was generalized by Austin and Virk in [1] into the notion of coarsely finite-to-1 maps, which are those maps $f : X \to Y$ between metric spaces such that for every $R > 0$ there exist $S > 0$ and $m \in \mathbb{N}$ so that the preimage of any subset C of Y with diameter at most R can be covered by at most m subsets of X with diameter at most S. In this paper we introduce an adjustment to this definition by giving the number $S > 0$ first: given an $S > 0$, we say that a map $f : X \to Y$ of metric spaces is *coarsely* S*-finite-to-*1 if for every $R > 0$ there exists an $m \in \mathbb{N}$ such that the preimage of any subset C of Y with diameter at most R can be covered by at most m subsets of X with diameter at most S. We then show that for a global morphism $f : (\Xi, \Xi^\infty) \to (\Lambda, \Lambda^\infty)$ of countable approximate groups, which is also proper, we have that $f : \Xi^\infty \to \Lambda^\infty$ is a coarsely $(\mathrm{diam}(\ker f))$-finite-to-1 map, and each k-component $f_k = f|_{\Xi^k} : \Xi^k \to \Lambda^k$ is a coarsely $(\mathrm{diam}(\ker f \cap \Xi^{2k}))$-finite-to-1 map.

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1 Introduction

The notion of an n-*to*-1 function, often used in topology, is referring to a function between topological spaces for which every point in codomain has at most n points contained in its preimage, that is, each fiber of this function contains at most n points.

A coarse geometry generalization of this notion is a *coarsely* n-*to*-1 *function*, introduced by T. Miyata and Ž. Virk in [7], defined as follows: given an $n \in \mathbb{N}$, a function $f : X \to Y$ of metric spaces is said to be *coarsely* n-*to*-1 if for every $R > 0$ there exists an $S > 0$ such that the preimage of any subset C of Y with diameter $\operatorname{diam} C \leq R$ can be covered by at most n subsets of X with diameter at most S.

In [1], K. Austin and Ž. Virk introduced the notion of a *coarsely finite-to*-1 *function*, which can be defined as follows: a function $f : X \to Y$ of metric spaces is said to be *coarsely finite-to*-1 if for every $R > 0$ there exist $S > 0$ and $m \in \mathbb{N}$ such that the preimage of any subset C of Y with diameter $\operatorname{diam} C \leq R$ can be covered by at most m subsets of X with diameter at most S. Note that in this definition both m and S may depend on R.

In this paper we introduce the notion of a *coarsely* S-*finite-to*-1 *function* between metric spaces, where the number $S > 0$ is predetermined and it does not depend on the choice of R: given an $S > 0$, we say that a function $f : X \to Y$ of metric spaces is *coarsely* S-*finite-to*-1 if for every $R > 0$ there exists an $m \in \mathbb{N}$ such that the preimage of any subset C of Y with diameter $\operatorname{diam} C \leq R$ can be covered by at most m subsets of X with diameter at most S. Such functions make sense in the context of homomorphisms between countable groups regarded as metric spaces, in particular when these homomorphisms are *proper* maps, i.e., when they have the property that the preimages of compact subsets of codomain are compact in domain. These homomorphisms have finite kernels, and kernels play a role in establishing the value of S in statements on coarse S-finite-to-1-ness of these maps. In fact, the role of $\ker f$ will be shown in Theorem 20, which is stated for global morphisms of countable approximate groups: this theorem says that when $f : (\Xi, \Xi^\infty) \to (\Lambda, \Lambda^\infty)$ is a global morphism of countable approximate groups which is also proper, then $f : \Xi^\infty \to \Lambda^\infty$ is a coarsely $(\operatorname{diam}(\ker f))$-

finite-to-1 map, and each k-component $f_k = f|_{\Xi^k} : \Xi^k \to \Lambda^k$ is a coarsely $(\operatorname{diam}(\ker f \cap \Xi^{2k}))$-finite-to-1 map. Although we are using the notion of countable approximate groups, we are treating these as metric spaces with left-invariant proper metrics, so the proof of Theorem 20 is easy to follow from the point of view of metric geometry.

Before reaching the proof of Theorem 20 in Section 4, we dedicate Section 3 to some basic facts on approximate groups and global morphisms between them. Section 2 contains notation and a reminder of some definitions needed in the rest of the paper.

2 Notation and some notions we will need

Besides notation, this section will contain some reminders regarding metric spaces, the properties of functions between them, and choosing "nice" metrics on countable groups.

Our notation $\mathbb{N}$ for natural numbers does not include zero, that is, $\mathbb{N} = \{1, 2, 3, \ldots\}$. For a set S, $|S|$ stands for the number of elements of S.

Furthermore, let us introduce some notation considering subsets of a group, so let A and B be subsets of a group $(G, \cdot)$. Then $AB := \{ab \mid a \in A, b \in B\}$, so $A^2 = AA = \{ab \mid a, b \in A\}$ and $A^k = A^{k-1}A$, for $k \in \mathbb{N}_{\geq 2}$. We use $A^{-1} := \{a^{-1} \mid a \in A\}$, and if $A = A^{-1}$, we say that A is *symmetric*. If $g \in G$, then $gA := \{ga \mid a \in A\}$. We mark the identity element of the group G by e or e_G.

Next we move on to metric spaces. Given a metric space (X, d), we denote by $B_d(x, r)$ the open ball, and by $\overline{B}_d(x, r)$ the closed ball in X centered at the point $x \in X$ and with radius $r > 0$. If there is only one metric we need, we will omit writing the index d from $B_d(x, r)$. Recall that a metric space (X, d) is *proper*, i.e., metric d is *proper*, if all closed balls (with bounded radii) in (X, d) are compact. Also, a metric d on a group G is *left-invariant* if $d(gh, gk) = d(h, k)$, for all $g, h, k \in G$.

It is easy to see that if G is a group with a left-invariant metric and $R > 0$, then for any $g \in G$ we have $\overline{B}(g, R) = g\overline{B}(e, R)$.

Since we will be focusing on countable groups, note that we will always consider countable groups as discrete groups, i.e., topological groups with discrete topology. In [5, Section 1] it is described how on a countable group (which need not be finitely generated) one can always choose a left-invariant proper metric so that this metric agrees with discrete topology on the group.

Therefore, whenever we need a metric on a countable group, we will always choose it to be a left-invariant proper metric. Note that when the metric is proper, the closed balls with respect to this metric in a countable group are compact and so they are finite sets (as subsets of a discrete space), which also means that open balls with bounded radii are finite sets.

Now we turn our attention to functions. Let us first remark that the word *map* is for us interchangeable with the word *function*, that is, a map need not be continuous. We already saw in the Introduction the definition of an n-to-1 function $f : X \to Y$ between sets X and Y as a function for which its fibers have at most n elements. Analogously, a function $f : X \to Y$ between sets is said to be *finite-to-*1 if the preimage of any point of Y is a finite set, i.e., if all fibers of f are finite.

Next we define a function being *proper* and *coarsely proper*, using definitions of these terms from [3] and [4] (notice that our coarsely proper function was called (just) proper in [2]).

Definition 1. *A function $f : X \to Y$ between metric spaces is called* proper *if preimages of compact sets are compact, and f is called* coarsely proper *if preimages of bounded sets are bounded.*

Let us state some facts connecting properness and coarse properness of a function between metric spaces.

Lemma 2. *If (X, d) is a metric space, (Y, d') is a proper metric space, and a map $f : X \to Y$ is proper, then f is coarsely proper.*

Proof. If D is a bounded set in Y, then D is contained in some closed ball $\overline{B}_{d'}(y, R)$ of Y. From Y being proper we get that $\overline{B}_{d'}(y, R)$ is compact in Y, so by properness of f we have that $f^{-1}(\overline{B}_{d'}(y, R))$ is compact in X. Since a compact set in a metric space must be bounded, we get that $f^{-1}(\overline{B}_{d'}(y, R))$ is bounded in X, so, as its subset, $f^{-1}(D)$ is also bounded in X. □

Lemma 3. *If (X, d) is a proper metric space with discrete topology induced by its metric, (Y, d') is a metric space and a map $f : X \to Y$ is coarsely proper, then f is proper.*

Proof. If C is a compact subset of Y, then C is bounded in Y, so since f is coarsely proper, $f^{-1}(C)$ is bounded in X, i.e., it is contained in some closed ball $\overline{B}_d(x, R)$ of X. From X being proper we get that $\overline{B}_d(x, R)$ is compact, and since (X, d) has discrete topology, $\overline{B}_d(x, R)$ must be a finite set, so its subset $f^{-1}(C)$ is also finite and therefore compact in X. □

Lemma 4. *If both* (X, d) *and* (Y, d') *are proper metric spaces with discrete topology induced by their metric, then the following statements are equivalent for a map* $f : X \to Y$*:*

(1) f *is coarsely proper,*

(2) f *is proper,*

(3) f *is finite-to-*1*.*

Proof. Lemmas 2 and 3 give us the equivalence of (1) and (2). We get (3) implying (2) from the finiteness of every compact subset of Y, and (2) implying (3) from the finiteness of every compact subset of X. □

Next we need to recall a well-known fact on fibers of group homomorphisms.

Lemma 5. *If* $f : G \to H$ *is an epimorphism of groups, then for any* $h \in H$ *we have* $f^{-1}(h) = g_h \ker f$*, where* g_h *is an element of* G *such that* $f(g_h) = h$*.*

If G is a countable group taken with a left-invariant proper metric, and H is any group taken with any metric, then for a group homomorphism $f : G \to H$ which is also a *proper* map all fibers are compact subsets of G, and therefore finite. Together with Lemma 5 this gives us:

Corollary 6. *Let* G *be a countable group with a left-invariant proper metric and let* H *be a group taken with any metric. If* $f : G \to H$ *is a group homomorphism which is proper, then* $\ker f$ *is finite and* f *is a* $|ker f|$*-to-*1 *map.*

3 Approximate groups and their global morphisms

This section contains some basic facts on approximate groups. Much more detailed account can be found in [3], but we will stick to only what is needed, which will be parallel to what was done in [6]. First we need to introduce the notion of an *approximate subgroup* of a group. The idea behind this is to spoil, in a controlled way, the fact that a subgroup is closed under the group operation, while keeping the properties of containing the inverse for each of its elements, and containing the identity element. The following definition is due to T. Tao ([8]):

Definition 7 (Approximate subgroup of a group)**.** *Let G be a group and let $k \in \mathbb{N}$. A subset $\Lambda \subset G$ is called a k-*approximate subgroup *of G if*

(1) $\Lambda = \Lambda^{-1}$ and $e \in \Lambda$, and

(2) there exists a finite subset $F \subset G$ such that $\Lambda^2 \subset \Lambda F$ and $|F| = k$.

We say that Λ is an approximate subgroup *of G if it is a k-approximate subgroup for some $k \in \mathbb{N}$.*

As long as the set F is finite, the number of its elements is not going to be important to us, since we will be interested in countably infinite approximate (sub)groups. It is clear from the definition that the result of the group operation between two elements of Λ is allowed to be outside of Λ, but it can only be "a finite set away". Note that for Λ there is the smallest subgroup $\Lambda^\infty := \bigcup_{k\in\mathbb{N}} \Lambda^k$ of G which contains Λ. We will use Λ^∞ to define:

Definition 8 (Approximate group)**.** *If Λ is an approximate subgroup of a group G, then the group $\Lambda^\infty = \bigcup_{k\in\mathbb{N}} \Lambda^k$, which is the smallest subgroup of G containing Λ, is called the* enveloping group *of Λ. The pair $(\Lambda, \Lambda^\infty)$ is called an* approximate group.

We say that an approximate group $(\Lambda, \Lambda^\infty)$ is *finite* if Λ is finite (but clearly Λ^∞ need not be finite). We say $(\Lambda, \Lambda^\infty)$ is *countable* if Λ is countable, which also implies that Λ^∞ is countable, and this is the case in which we are particularly interested.

The reason why we refer to $(\Lambda, \Lambda^\infty)$ instead of just Λ is in the fact that we need to regard Λ and Λ^k for any $k \in \mathbb{N}$ as metric spaces, and we do so by introducing a (left-invariant proper) metric on Λ^∞, so in fact, Λ^∞ is the ambient metric space for Λ^k, and therefore should be prominent.

Here are some basic examples and a non-example of approximate (sub)groups, taken from [3] and also mentioned in [6]:

Example 9 (A non-example)**.** *In $(\mathbb{Z}, +)$ define $\Lambda := \{2^i \mid i \in \mathbb{Z}\} \cup \{0\} \cup \{-2^i \mid i \in \mathbb{Z}\}$, which is clearly a symmetric set with 0. However, Λ is not an approximate subgroup of $\mathbb{Z}$, because $\Lambda + \Lambda$ contains $2^n + 2^{n+1} = 3 \cdot 2^n$ for each $n \in \mathbb{N}$, which are not in Λ, and it is easy to see that there is no finite set $F \subseteq \mathbb{Z}$ such that $\Lambda + \Lambda \subseteq \Lambda + F$.*

Example 10. *In any group G, any subgroup H is obviously an approximate subgroup of G, and since $H^\infty = H$, the pair (H, H) is an approximate group.*

If F is a finite symmetric subset of G and $e \in F$, then F is clearly an approximate subgroup of G, so (F, F^∞) is an approximate group. If Λ is an approximate subgroup of a group G, then Λ^k is also an approximate subgroup of G, so $(\Lambda^k, \Lambda^\infty)$ is an approximate group, for all $k \in \mathbb{N}$.

Example 11. *Let $\mathrm{BS}(1,2) = \langle a, b \mid bab^{-1} = a^2 \rangle$, i.e., the Baumslag-Solitar group of type $(1,2)$, and define $\Lambda := \langle a \rangle \cup \{b, b^{-1}\}$. Then Λ is symmetric, containing identity and $\Lambda^\infty = \mathrm{BS}(1,2)$. Using the defining relation and $(b^{-1}ab)^2 = a$ it can be shown that $\Lambda^2 \subseteq \Lambda \cdot \{e, b, b^{-1}, b^{-1}a\}$, hence $(\Lambda, \Lambda^\infty)$ is an approximate group.*

Here is a more demanding example coming from lattice theory, quoted from [3, Example 2.87]):

Example 12 (Cut-and-project construction)**.** *Let G and H be locally compact groups, and denote by $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ the canonical projections. Let Γ be a subgroup of $G \times H$ such that the restriction $\pi_G|_\Gamma$ is injective. Then for any relatively compact symmetric identity neighborhood W in H, the set $\Lambda(\Gamma, W) := \pi_G(\Gamma \cap (G \times W))$ is an approximate subgroup of G, so $(\Lambda(\Gamma, W), (\Lambda(\Gamma, W))^\infty)$ is an approximate group. This $\Lambda(\Gamma, W)$ is referred to as a* cut-and-project set*, because it arises from Γ by first cutting it with the "strip" $G \times W$ in $G \times H$, and then projecting down to G.*

Using homomorphisms between the enveloping groups, next we introduce the *global morphisms* of approximate groups:

Definition 13. *Let (Ξ, Ξ^∞) and $(\Lambda, \Lambda^\infty)$ be approximate groups. A* global morphism *$f : (\Xi, \Xi^\infty) \to (\Lambda, \Lambda^\infty)$ is a group homomorphism $f : \Xi^\infty \to \Lambda^\infty$ for which $f(\Xi^k) \subseteq \Lambda^k$, for all $k \in \mathbb{N}$.*

*Each restriction $f_k := f|_{\Xi^k} : \Xi^k \to \Lambda^k$ is referred to as the k-*component *of f, and it is a so-called* partial homomorphism*, which means that whenever a, b and ab are in Ξ^k, then $f_k(ab) = f_k(a) f_k(b)$.*

It is easy to see that for a global morphism $f : (\Xi, \Xi^\infty) \to (\Lambda, \Lambda^\infty)$, $f(\Xi)$ is an approximate subgroup of Λ^∞ (see [3, Example 2.51]), so in the proofs that follow, we will often decide to replace $(\Lambda, \Lambda^\infty)$ with $(f(\Xi), f(\Xi)^\infty)$, or just ask for a global morphism to be surjective.

Let us also note here that if we ask for a global morphism between approximate groups to also be a proper map, this will mean we are asking for $f : \Xi^\infty \to \Lambda^\infty$ to be proper.

4 Global morphisms of countable approximate groups and coarse S-finite-to-1-ness

This final section is dedicated to proving Theorem 20, about global morphisms which are also proper, between countable approximate groups, and their properties of being coarsely diam(ker f)-finite-to-1, as well as their k-components being coarsely diam((ker f) $\cap\,\Xi^{2k}$)-finite-to-1. We begin by stating a technical lemma, helpful in future calculations.

Lemma 14. *Let $(\Lambda, \Lambda^\infty)$ be a countable approximate group with a left-invariant proper metric on Λ^∞ and let $k \in \mathbb{N}$. Then:*

(1) For any subset $A \subset \Lambda^\infty$ and any $x \in \Lambda^k$, we have $(xA) \cap \Lambda^k \subseteq x(A \cap \Lambda^{2k})$.

(2) For any $x \in \Lambda^k$ and any $R > 0$ we have $|\overline{B}(x,R) \cap \Lambda^k| \leq |\overline{B}(e,R) \cap \Lambda^{2k}|$.

Proof. For the proof of (1), let $A \subseteq \Lambda^\infty$ and take any $x \in \Lambda^k$. Then for any $z \in (xA) \cap \Lambda^k$ there is a $w \in A$ such that $z = xw$, and also there is an $\tilde{x} \in \Lambda^k$ so that $z = \tilde{x}$. Therefore from $\tilde{x} = xw$ it follows that $w = x^{-1}\tilde{x} \in \Lambda^{2k}$, so $z \in x(A \cap \Lambda^{2k})$.

For the proof of (2), let $R > 0$ and take any $x \in \Lambda^k$. Apply (1) to $\overline{B}(x,R) = x\overline{B}(e,R)$ to get

$$\overline{B}(x,R) \cap \Lambda^k = (x\overline{B}(e,R)) \cap \Lambda^k \overset{(1)}{\subseteq} x(\overline{B}(e,R) \cap \Lambda^{2k}).$$

Therefore $|\overline{B}(x,R) \cap \Lambda^k| \leq |\overline{B}(e,R) \cap \Lambda^{2k}|$. □

Lemma 15. *Let (Ξ, Ξ^∞) and $(\Lambda, \Lambda^\infty)$ be countable approximate groups and let $f : (\Xi, \Xi^\infty) \to (\Lambda, \Lambda^\infty)$ be a global morphism which is proper. Then $f : \Xi^\infty \to \Lambda^\infty$ is a $|\ker f|$-to-1 map. Moreover, for each $k \in \mathbb{N}$, the k-component $f_k = f|_{\Xi^k} : \Xi^k \to \Lambda^k$ is a $|(\ker f) \cap \Xi^{2k}|$-to-1 map.*

Proof. Take some left-invariant proper metrics on Ξ^∞ and Λ^∞ and, for the sake of simplicity, assume that $f : (\Xi, \Xi^\infty) \to (\Lambda, \Lambda^\infty)$ is surjective as a map of pairs, so, in particular, each $f_k = f|_{\Xi^k} : \Xi^k \to \Lambda^k$ is surjective, for all $k \in \mathbb{N}$.

The statement that $f : \Xi^\infty \to \Lambda^\infty$ is a $|\ker f|$-to-1 map is true by Corollary 6.

To show the statement for components of f, let $k \in \mathbb{N}$ and let λ be a random element from Λ^k. Since $f_k : \Xi^k \to \Lambda^k$ is surjective, for this $\lambda \in \Lambda^k$ there is a $\xi_\lambda \in \Xi^k$ such that $f_k(\xi_\lambda) = \lambda$. Therefore, using Lemma 5 and Lemma 14 (1) we get:

$$f_k^{-1}(\lambda) = f^{-1}(\lambda) \cap \Xi^k = (\xi_\lambda \ker f) \cap \Xi^k \subseteq \xi_\lambda((\ker f) \cap \Xi^{2k}),$$

so $|f_k^{-1}(\lambda)| \leq |\xi_\lambda((\ker f) \cap \Xi^{2k})| = |(\ker f) \cap \Xi^{2k}|$, which finishes the proof. □

Remark 16. *The estimate* $|f_k^{-1}(\lambda)| \leq |(\ker f) \cap \Xi^{2k}|$ *may seem too high, since we know that* $|\xi_\lambda \ker f| = |\ker f|$, *but note that comparing the number of elements in* $\xi_\lambda \ker f$ *and* $\ker f$ *within the approximate subgroup* Ξ^k *is not the same as comparing* $\xi_\lambda \ker f$ *and* $\ker f$ *within the group* Ξ^∞. *And we need an estimate independent of* ξ_λ, *so the price of using the formula (1) of Lemma 14 and pushing* ξ_λ *in front of the expression is doubling of the power from* Ξ^k *to* Ξ^{2k}.

Now let us state definitions for coarsely n-to-1 maps, coarsely finite-to-1 maps and coarsely S-finite-to-1 maps that we mentioned in the Introduction, which translate the concepts of n-to-1 maps and finite-to-1 maps to coarse geometry. We start with the notion introduced in [7], and rephrased in [1]:

Definition 17. *Let* $n \in \mathbb{N}$. *A function* $f : X \to Y$ *of metric spaces is said to be* coarsely n-to-1 *if for every* $R > 0$ *there exists an* $S > 0$ *such that the preimage of each subset* C *of* Y *with diameter* $\operatorname{diam} C \leq R$ *can be covered by at most* n *subsets of* X *with diameter at most* S.

Next, the notion introduced in [1] and restated for metric spaces in [2, Definition 5.12]:

Definition 18. *A function* $f : X \to Y$ *of metric spaces is said to be* coarsely finite-to-1 *if for every* $R > 0$ *there exist* $S > 0$ *and* $m \in \mathbb{N}$ *such that the preimage of each subset* C *of* Y *with diameter* $\operatorname{diam} C \leq R$ *can be covered by at most* m *subsets of* X *with diameter at most* S.

Note that in the previous definition both m and S (may) depend on R. We introduce a change in Definition 18 so that S is given first and it does not depend on the choice of R.

Definition 19. *Let* $S > 0$. *A function* $f : X \to Y$ *of metric spaces is said to be* coarsely S-finite-to-1 *if for every* $R > 0$ *there exists an* $m \in \mathbb{N}$ *such that the preimage of each subset* C *of* Y *with diameter* $\operatorname{diam} C \leq R$ *can be covered by at most* m *subsets of* X *with diameter at most* S.

Now we are ready to prove our main result:

Theorem 20. *Let* (Ξ, Ξ^∞) *and* $(\Lambda, \Lambda^\infty)$ *be countable approximate groups, and let* $f : (\Xi, \Xi^\infty) \to (\Lambda, \Lambda^\infty)$ *be a global morphism which is also proper. Then* $f : \Xi^\infty \to \Lambda^\infty$ *is a coarsely* $(\operatorname{diam}(\ker f))$*-finite-to-*1 *map. Moreover, for every* $k \in \mathbb{N}$*, the* k*-component* $f_k = f|_{\Xi^k} : \Xi^k \to \Lambda^k$ *is a coarsely* $(\operatorname{diam}(\ker f \cap \Xi^{2k}))$*-finite-to-*1 *map.*

Proof. Let us start by taking d on Ξ^∞ and d' on Λ^∞ to be some left-invariant proper metrics, and recall that $\overline{B}_{d'}(\lambda, R)$ is notation for a closed ball in (Λ^∞, d'). We may assume that $f : (\Xi, \Xi^\infty) \to (\Lambda, \Lambda^\infty)$ is surjective as a map of pairs, so, in particular, each $f_k = f|_{\Xi^k} : \Xi^k \to \Lambda^k$ is surjective, for all $k \in \mathbb{N}$.

Now take an $R > 0$ and fix it. To begin with, let $C \subseteq \Lambda^\infty$ be any subset with $\operatorname{diam}_{d'} C \leq R$. Then there is a $\lambda \in C$ such that $C \subseteq \overline{B}_{d'}(\lambda, R)$. Recall that $\overline{B}_{d'}(\lambda, R) = \lambda \overline{B}_{d'}(e_\Lambda, R)$, and put $m := |\overline{B}_{d'}(e_\Lambda, R)|$, which we know is finite since the metric d' is proper, and it depends on R, but does not depend on λ. Therefore $\overline{B}_{d'}(\lambda, R)$ also has m elements, so let us write $\overline{B}_{d'}(\lambda, R) = \{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda^\infty$. Since f is surjective, pick some elements $\xi_i \in \Xi^\infty$ so that $f(\xi_i) = \lambda_i$, for $i = 1, \ldots, m$. Then

$$f^{-1}(C) \subseteq f^{-1}(\overline{B}_{d'}(\lambda, R)) = \bigsqcup_{i=1}^{m} f^{-1}(\lambda_i) = \bigsqcup_{i=1}^{m} \xi_i \ker f,$$

and by left-invariance of the metric d on Ξ^∞ we have that $\operatorname{diam}_d(\xi_i \ker f) = \operatorname{diam}_d(\ker f)$, for all i. Thus $f^{-1}(C)$ is covered by m sets of diameter $\operatorname{diam}_d(\ker f)$, so the first statement holds.

For the second statement, fix a $k \in \mathbb{N}$ and an $R > 0$. Take any subset $C \subseteq \Lambda^k$ with $\operatorname{diam}_{d'} C \leq R$, so there is a $\lambda \in C$ such that $C \subseteq \overline{B}_{d'}(\lambda, R) \cap \Lambda^k$. By Lemma 14 (2) we know that $|\overline{B}_{d'}(\lambda, R) \cap \Lambda^k| \leq |\overline{B}_{d'}(e_\Lambda, R) \cap \Lambda^{2k}| =: m \in \mathbb{N}$ (where this m does not depend on λ). Therefore we can write $\overline{B}_{d'}(\lambda, R) \cap \Lambda^k = \{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$, where $\ell \leq m$, and, because $f_k : \Xi^k \to \Lambda^k$ is surjective, we can choose some $\xi_i \in \Xi^k$ so that $f_k(\xi_i) = \lambda_i$, for $i = 1, \ldots, \ell$. Then, using

Lemma 14 (1) we have $(\xi_i \ker f) \cap \Xi^k \subseteq \xi_i(\ker f \cap \Xi^{2k})$, so

$$\begin{aligned} f_k^{-1}(C) = f^{-1}(C) \cap \Xi^k &\subseteq f^{-1}(\overline{B}_{d'}(\lambda, R) \cap \Lambda^k) \cap \Xi^k \\ &= \bigsqcup_{i=1}^{\ell} f^{-1}(\lambda_i) \cap \Xi^k = \bigsqcup_{i=1}^{\ell} (\xi_i \ker f) \cap \Xi^k \\ &\subseteq \bigsqcup_{i=1}^{\ell} \xi_i(\ker f \cap \Xi^{2k}). \end{aligned}$$

By left-invariance of the metric d on Ξ^∞ we get

$$\mathrm{diam}_d(\xi_i(\ker f \cap \Xi^{2k})) = \mathrm{diam}_d(\ker f \cap \Xi^{2k}),$$

so $f_k^{-1}(C)$ can be covered by at most m sets of diameter $\mathrm{diam}_d(\ker f \cap \Xi^{2k})$. □

Remark 21. *As in Remark 16, estimating diameters of* $(\xi_i \ker f) \cap \Xi^k$ *by diameters of* $\ker f \cap \Xi^{2k}$ *might seem like overkill, but again, comparing* $\xi_i \ker f$ *and* $\ker f$ *within the approximate subgroup* Ξ^k *is not the same as comparing* $\xi_i \ker f$ *and* $\ker f$ *within the group* Ξ^∞.

Remark 22. *By Lemma 4, instead of asking that the global morphism in Lemma 15 and Theorem 20 be proper, we could have asked for it to be coarsely proper, or finite-to-*1.

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