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Class graphs obtained from residual designs of new symmetric (71,15,3) designs

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Abstract

It is known that a residual design of a symmetric $(71, 15, 3)$ design that satisfies certain conditions leads to a strongly regular graph with parameters $(35, 16, 6, 8)$, called a class graph. It is established in [5], [6], [7] and [3] that the 148 symmetric $(71, 15, 3)$ designs that were known until then produce exactly six class graphs. We show that 22 symmetric $(71, 15, 3)$ designs constructed in [4] lead to 344 new residual designs with parameters 2-(56,12,3), that produce five pairwise non-isomorphic class graphs. The corresponding class graphs are isomorphic to the previously known class graphs, so the 170 known symmetric $(71, 15, 3)$ designs produce exactly six class graphs being strongly regular graphs with parameters $(35, 16, 6, 8)$.

Keywords: block design, residual design, class graph
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1 Introduction and preliminaries

A design $\mathcal{D}$ with parameters t-(v,k,λ) is a finite incidence structure $(\mathcal{P},\mathcal{B},\mathcal{I})$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $\mathcal{I}\subseteq\mathcal{P}\times\mathcal{B}$, with the following properties:

1. $|\mathcal{P}|=v$ and $1<k<v-1$,
2. every element (block) of $\mathcal{B}$ is incident with exactly k elements (points) of $\mathcal{P}$,
3. every t distinct points in $\mathcal{P}$ are together incident with exactly λ blocks of $\mathcal{B}$.

If a design is simple, i.e. does not have repeated blocks, then we can identify blocks with subsets of the point set $\mathcal{P}$ in a natural way. A simple design is called complete if it has $\binom{v}{k}$ blocks, otherwise it is called incomplete. A balanced incomplete block design (BIBD) is an incomplete design with $t=2$. The number of blocks in a block design is denoted by b. Each point is contained in exactly $r=\frac{\lambda(v-1)}{k-1}$ blocks. If $v=b$ (equivalently, $r=k$), a design is called symmetric.

An isomorphism from one design to another is a bijective mapping of points to points and blocks to blocks which preserves incidence. An isomorphism from a design $\mathcal{D}$ onto $\mathcal{D}$ is called an automorphism of $\mathcal{D}$. The set of all automorphism of the design $\mathcal{D}$ is a group called the full automorphism group of $\mathcal{D}$, denoted by $\mathrm{Aut}(\mathcal{D})$. Each subgroup of the $\mathrm{Aut}(\mathcal{D})$ is called an automorphism group of $\mathcal{D}$.

For a symmetric (v,k,λ)-BIBD $\mathcal{D}=(\mathcal{P},\mathcal{B},\mathcal{I})$, design

$$\mathrm{Res}(\mathcal{D},B_0)=(\mathcal{P}\backslash B_0,\{B\backslash B_0|B\in\mathcal{B},B\neq B_0\},\mathcal{I})$$

is a residual design with respect to the block B_0. $\mathrm{Res}(\mathcal{D},B_0)$ is a $(v-k,k-\lambda,\lambda)$-BIBD.

Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{I})$ be a simple k-regular graph with v vertices. $\mathcal{G}$ is strongly regular with parameters (v,k,λ,μ) if every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have μ common neighbors. An isomorphism from a graph $\mathcal{G}_1$ to a graph $\mathcal{G}_2$ is a bijection from the set of vertices of $\mathcal{G}_1$ onto the set of vertices of $\mathcal{G}_2$ that preserves adjacency. An isomorphism from a graph $\mathcal{G}$ to itself is called an automorphism of $\mathcal{G}$. The set of all automorphisms of $\mathcal{G}$ is called a full automorphism group of $\mathcal{G}$ and it denoted by $\mathrm{Aut}(\mathcal{G})$

In [7], it was shown that there exist 1122 pairwise non-isomorphic 2-$(56, 12, 3)$ designs being the residual designs of the 146 symmetric $(71, 15, 3)$ designs given in [5] and [6]. Furthermore, 2 new symmetric $(71, 15, 3)$ designs were constructed from codes in [3]. They yield 30 pairwise non-isomorphic 2-$(56, 12, 3)$ residual designs.

Since then, 22 new symmetric $(71, 15, 3)$ designs were constructed using a genetic algorithm in [4]. We refer to the designs constructed in [4] as the new symmetric $(71, 15, 3)$ designs.

Let $\mathcal{D}$ be a (v, k, λ)-BIBD with exactly three distinct intersection numbers $k - r + \lambda$, ρ_1 and ρ_2, where $\rho_1 > \rho_2$. In this case, as shown in [5], a strongly regular graph can be constructed from this design and it is called the class graph of $\mathcal{D}$. Two blocks B_1 and B_2 of the design $\mathcal{D}$ are equivalent if $|B_1 \cap B_2| \in \{k, k - r + \lambda\}$ (see [1]). A class graph of $\mathcal{D}$ is a graph whose vertices are equivalence classes and two vertices are adjacent if two blocks representing the corresponding classes have ρ_1 points in common.

For the computations in this paper we used programs written in GAP [8].

2 (56,12,3)-BIBDs

Let $\mathcal{D}$ be a symmetric design and let B_0 and B_1 be blocks of $\mathcal{D}$ belonging to the same orbit of $\mathrm{Aut}(\mathcal{D})$. It is shown in [2, Corollary 1] that the residual designs with respect to the blocks B_0 and B_1 are isomorphic. Hence, to construct all residual designs of $\mathcal{D}$, up to isomorphism, it is sufficient to construct residual designs with respect to representatives of the $\mathrm{Aut}(\mathcal{D})$-orbits.

The 22 symmetric $(71, 15, 3)$ designs constructed in [4] yield 344 pairwise non-isomorphic (56,12,3)-BIBDs. Including 1122 designs from [7] and 30 designs from [3], this gives 1496 (56,12,3)-BIBDs out of which 1495 are pairwise non-isomorphic. We give the information about these 1495 designs in Table 1.

3 Class graphs of (56,12,3)-BIBDs

The 148 symmetric $(71, 15, 3)$ designs produce exactly six class graphs, as it is established in [5], [6], [7] and [3]. We present the information about these graphs in Table 2.

$\|\mathrm{Aut}(\mathcal{D})\|$	$\mathrm{Aut}(\mathcal{D})$ structure	number of designs
336	$(E_8 : F_{21}) \times Z_2$	2
168	$E_8 : F_{21}$	1
48	$E_4 \times A_4$	18
42	$F_{21} \times Z_2$	6
24	$E_4 \times S_3$	12
24	$A_4 \times Z_2$	137
21	F_{21}	1
16	E_{16}	61
12	D_{12}	32
12	A_4	20
8	E_8	223
6	Z_6	120
4	E_4	210
3	Z_3	101
2	Z_2	377
1	I	174

Table 1: 1495 pairwise non-isomorphic (56,12,3)-BIBDs

$\|\mathrm{Aut}(\mathcal{G})\|$	$\mathrm{Aut}(\mathcal{G})$ structure	number of graphs
40320	S_8	1
288	$(A_4 \times A_4) : Z_2$	1
192	$((E_8 : E_4) : Z_3) : Z_2$	1
96	$(E_{16} : Z_2) : Z_3$	1
32	$E_{16} : Z_2$	1
12	A_4	1

Table 2: Six pairwise non-isomorphic graphs obtained from residual designs of the 148 symmetric (71,15,3) designs given in [5], [6], [7] and [3]

The 344 pairwise non-isomorphic (56,12,3)-BIBDs obtained from 22 new symmetric $(71, 15, 3)$ designs [4] have intersection numbers of blocks $\{0, 1, 2, 3\}$, $\{0, 2, 3\}$ and $\{1, 2, 3\}$. Since $r = \frac{\lambda(v-1)}{k-1} = 15$ and $k - r + \lambda = 12 - 15 + 3 = 0$, we are interested in intersection numbers $\{0, 2, 3\}$, where $\rho_1 = 3$, $\rho_2 = 2$.

Among 344 designs yielded from [4] there are 28 designs with intersection numbers $\{0, 2, 3\}$. According to [5], for each of those 28 designs it is possible to construct the corresponding class graph, being a strongly regular graph on 35 vertices, whose vertices are equivalence classes (two blocks B_1 and B_2 are equivalent if $|B_1 \cap B_2| = 0$), two vertices being adjacent if two blocks representing the corresponding classes have $\rho_1 = 3$ points in common.

We obtain five pairwise non-isomorphic strongly regular graphs with parameters $(35, 16, 6, 8)$. Each of these strongly regular graphs is isomorphic to one of the graphs from Table 2 with full automorphism groups of orders 40320, 288, 192, 32 and 12.

4 Conclusion

The 22 new symmetric (71,15,3) designs from [4] do not lead to new class graphs. Hence, up to isomorphism there are exactly six strongly regular graphs with parameters $(35, 16, 6, 8)$ that can be constructed as class graphs of the 170 known symmetric $(71, 15, 3)$ designs.

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