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A note on Banach fixed point property

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Abstract

In this short note we consider a sort of converse of the Banach fixed point theorem and prove that a metric space X is complete if and only if, for each closed subspace $Y \subseteq X$, any contraction $f\colon Y \to Y$ has a fixed point $y \in Y$.

Keywords: complete metric space, contraction, fixed point, Banach fixed point theorem, Banach fixed point property
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1 Introduction and main result

A *metric space* is an ordered pair (X, d), where X is a nonempty set and d is a metric (distance function) on X. A map $f\colon X \to X$ from a metric space (X, d) into itself is called a *contraction*, if there exists a coefficient $\kappa \in [0, 1\rangle$ such that $d(f(x), f(x')) \leq \kappa d(x, x')$, for any $x, x' \in X$. The remarkable Banach fixed point theorem states that any contraction $f\colon X \to X$ from a complete metric space (X, d) has a (unique) fixed point $y \in X$. Even more, it gives a procedure, a constructive method, for reaching the fixed point y of f and, for each step of the procedure, estimates the "error", the distance to the fixed point y. Precisely, for an arbitrary point $x \in X$, a sequence (x_n) in X recursively defined by $x_1 = x$ and $x_n = f(x_{n-1})$, for $n \geq 2$, converges to the fixed point y and $d(x_n, y) \leq \frac{\kappa^n}{1-\kappa} d(x, f(x))$, $n \in \mathbb{N}$, where κ denotes the coefficient of the contraction f (see [1, Theorem 7.2]). The property of contractions given in the Banach fixed point theorem was a motivation for introducing the following notion.

Definition 1. *A metric space* (X, d) *is said to satisfy the Banach fixed point property* ($BFPP$, *for short*) *if every contraction* $f\colon X \to X$ *has a fixed point.*

So, the Banach fixed point theorem states that a complete metric space satisfies $BFPP$. Then a natural question arises: If a metric space (X, d) has $BFPP$, is (X, d) complete? In other words, is the converse of the Banach fixed point theorem true? As M. Elekes shows in his paper [2], it is known for a long time that the answer to this question is negative and the following counterexample, presented by E. Behrends in 2006, was considered as "folklore". Let $X = \big\{(x, \sin\frac{1}{x})\colon x \in \langle 0, 1]\big\} \subseteq \mathbb{R}^2$ be a subspace of the Euclidean plane $(\mathbb{R}^2, d_2)$, where d_2 denotes the Euclidean metric. Obviously, X is not complete, but it satisfies $BFPP$. M. Elekes provides a simple proof that X satisfies $BFPP$ ([2, Theorem 1.2]). The proof is based on a fact that, for each contraction $f\colon X \to X$, there exists $a, 0 < a < 1$, such that $f(X) \subseteq \big\{(x, \sin\frac{1}{x})\colon x \in [a, 1]\big\}$. Since a subset $X_{[a,1]} = \big\{(x, \sin\frac{1}{x})\colon x \in [a, 1]\big\} \subseteq X$ is compact, $X_{[a,1]}$ is complete and applying the Banach fixed point theorem to the restriction $f\,\big|X_{[a,1]}$ of f to $X_{[a,1]}$ we get a fixed point $y \in X_{[a,1]}$ for f. However, studying the space X we noticed the following interesting phenomenon.

Proposition 2. *A metric space* $X = \big\{(x, \sin\frac{1}{x})\colon x \in \langle 0, 1]\big\} \subseteq \mathbb{R}^2$ *admits a closed subspace* $Y \subseteq X$ *which does not satisfy* $BFPP$.

Proof. Put $Y = \big\{(\frac{1}{n\pi}, \sin n\pi)\colon n \in \mathbb{N} \setminus \{1,2\}\big\} = \big\{(\frac{1}{n\pi}, 0)\colon n \in \mathbb{N} \setminus \{1,2\}\big\} \subseteq X$. Obviously, Y is closed in X since the closure $\mathrm{Cl}\, Y$ of Y in $\mathbb{R}^2$ equals $\mathrm{Cl}\, Y = Y \cup \{(0,0)\}$ and then $\mathrm{Cl}_X Y = X \cap \mathrm{Cl}\, Y = Y$. Now, we define a map $f\colon Y \to Y$ by $f(\frac{1}{n\pi}, 0) = \big(\frac{1}{(n^2+1)\pi}, 0\big)$. It is clear that f does not have a fixed point. We claim that f is a contraction with the coefficient $\kappa = \frac{2}{3}$. Take arbitrary $n, k \in \mathbb{N} \setminus \{1,2\}$. Note that $d_2\left((\frac{1}{n\pi}, 0), (\frac{1}{k\pi}, 0)\right) = \left|\frac{1}{n\pi} - \frac{1}{k\pi}\right| = \frac{|k-n|}{nk\pi}$. Now consider $d_2\big(f(\frac{1}{n\pi}, 0), f(\frac{1}{k\pi}, 0)\big) = d_2\big((\frac{1}{(n^2+1)\pi}, 0), (\frac{1}{(k^2+1)\pi}, 0)\big) = \left|\frac{1}{(n^2+1)\pi} - \frac{1}{(k^2+1)\pi}\right|$ $= \frac{|k^2-n^2|}{(n^2+1)(k^2+1)\pi} = \frac{|k-n|(k+n)}{(n^2+1)(k^2+1)\pi}$. Since $n, k \geq 3$, it follows that $\frac{k+n}{nk} = \frac{1}{n} + \frac{1}{k} \leq \frac{2}{3}$. We get $d_2\left(f(\frac{1}{n\pi}, 0), f(\frac{1}{k\pi}, 0)\right) = \frac{|k-n|(k+n)}{(n^2+1)(k^2+1)\pi} \leq \frac{|k-n|(k+n)}{n^2k^2\pi} = \frac{|k-n|}{nk\pi}\frac{k+n}{nk} \leq \frac{2}{3} d_2\left((\frac{1}{n\pi}, 0), (\frac{1}{k\pi}, 0)\right)$, which proves that f is a contraction and, consequently, Y does not satisfy $BFPP$. $\square$

Let us conclude. An incomplete metric space $X = \big\{(x, \sin\frac{1}{x})\colon x \in \langle 0, 1]\big\} \subseteq \mathbb{R}^2$ satisfies $BFPP$ but admits an infinite closed proper subspace

$Y \subsetneq X$ which does not satisfy $BFPP$. It turned out that this observation led us to the following characterization theorem for completeness of metric spaces, our main result.

Theorem 3. *A metric space (X, d) is complete if and only if each closed subspace $Y \subseteq X$ of X satisfies $BFPP$.*

2 Proof of Theorem 3

Let us first note that Theorem 3 is true, but not of particular interest, if (X, d) is a finite metric space. Indeed, the following claim is obvious.

Proposition 4. *Let X be a nonempty finite set and let d be an arbitrary metric on X. Then (X, d) is a complete metric space.*

Proof. Since X is finite, (X, d) is a compact metric space and, consequently, complete (and totally bounded). Also note that the induced metric topology is discrete. □

Recall that a metric space (X, d) is totally bounded if and only if each sequence (x_n) in X admits a Cauchy subsequence (x_{n_k}). Our proof of Theorem 3 is based on the following characterization theorem for infinite metric spaces.

Theorem 5. *An infinite metric space (X, d) is complete if and only if each infinite totally bounded subset $A \subseteq X$ has an accumulation point x_0 in X.*

Proof. Let (X, d) be complete and assume that a subset $A \subseteq X$ is infinite and totally bounded. Then there exists in A a Cauchy sequence (a_n) whose terms are mutually different points of A. The sequence (a_n) is obtained in the following way. First, since A is infinite there exists an injective sequence (x_n) in A. So, all terms of (x_n) are mutually different points of A. On the other hand, since A is totally bounded, (x_n) admits a Cauchy subsequence (x_{n_k}). Putting $a_k = x_{n_k}, k \in \mathbb{N}$, we get the desired sequence (a_n). By assumption, (X, d) is complete, so (a_n) converges to some point $x_0 \in \operatorname{Cl} A \subseteq X$. Since $x_0 = \lim(a_n)$ and (a_n) consists of different points of A, it is obvious that each ball $B(x_0, \varepsilon)$ intersects $A \setminus \{x_0\}$. This shows that x_0 is an accumulation point of A.

Conversely, assume that each infinite totally bounded set $A \subseteq X$ has an accumulation point in X. Take an arbitrary Cauchy sequence (x_n) in X. Put $A = \{x_n \colon n \in \mathbb{N}\}$. If A is a finite set, there exists a point $x_0 \in X$ such that $x_n = x_0$ for infinitely many $n \in \mathbb{N}$. So, (x_n) admits a stationary subsequence (x_{n_k}). Since (x_n) is a Cauchy sequence and admits a convergent subsequence (x_{n_k}), it follows that (x_n) is a convergent sequence and converges to x_0. Now, assume that $A = \{x_n : n \in \mathbb{N}\}$ is an infinite set. For each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\{x_n : n \geq k\}$ is contained in a ball $B(x_k, \varepsilon)$, which implies that A is totally bounded. By assumption, there exists an accumulation point $x_0 \in X$ of the set A. Since every ball $B(x_0, \varepsilon)$ contains an infinitely many elements of A, it is easy to see that x_0 is an accumulation point of the sequence (x_n). Hence there is a subsequence (x_{n_k}) of the sequence (x_n) which converges to x_0. So, we conclude that (x_n) is convergent and converges to x_0. In both cases the sequence (x_n) is convergent, which shows that (X, d) is a complete space. □

In the proof of Theorem 3 we will need the following simple lemma.

Lemma 6. *Let (x_n) be a Cauchy sequence in a metric space (X, d) and let (ε_k) be a sequence of positive real numbers. Then there exists a subsequence (x_{n_k}) of the sequence (x_n) such that $n_k > k$ and $d(x_{n_j}, x_{n_k}) < \varepsilon_k$, for each $j \geq k$ and each $k \in \mathbb{N}$.*

Proof. The desired subsequence (x_{n_k}) will be obtained inductively. Precisely, by induction we will determine strictly increasing sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of positive integers such that $n_k > k$ and $d(x_n, x_{n_k}) < \varepsilon_k$, for each $n \geq n_k$ and each $k \in \mathbb{N}$.
Let $k = 1$. Since (x_n) is a Cauchy sequence, for the given $\varepsilon_1 > 0$, there is an $n' \in \mathbb{N}, n' > 1$, such that $d(x_n, x_m) < \varepsilon_1$ holds, for any $n, m \geq n'$. Put $n_1 := n'$ and we get $d(x_n, x_{n_1}) < \varepsilon_1$, for any $n \geq n_1 > 1$.
Assumed that $n_1 < n_2 < \cdots < n_k$ with the required properties are already determined. Then, for the given $\varepsilon_{k+1} > 0$, there exists an $n' \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon_{k+1}$ holds, for any $n, m \geq n'$. Put $n_{k+1} := \max\{n', n_k + 1\}$. Obviously, $n_{k+1} > n_k$ and $d(x_n, x_{n_{k+1}}) < \varepsilon_{k+1}$, for each $n \geq n_{k+1}$. Moreover, $n_{k+1} > n_k > k$ and $n_{k+1} > k + 1$ holds. Thus, the inductive step is proven.
The obtained subsequence (x_{n_k}) is the desired one. Indeed, for each $k \in \mathbb{N}$ and each $j > k$, $n_j > n_k$ holds and consequently $d(x_{n_j}, x_{n_k}) < \varepsilon_k$, by the properties of the inductive construction. □

Now, we are prepared to prove our main result.

Proof. (of Theorem 3). If (X, d) is complete and $Y \subseteq X$ is a closed subset of X, then the subspace (Y, d) is complete as well. Applying the Banach fixed point theorem to (Y, d), it follows that Y satisfies $BFPP$.

Conversely, assume that each closed subspace $Y \subseteq X$ of X satisfies $BFPP$. If X is finite, then X is complete according to Proposition 4. So, let us consider the case when X is an infinite set. According to Theorem 5, it is sufficient to prove that any infinite totally bounded subset $A \subseteq X$ has an accumulation point in X. So, let A be an arbitrary infinite totally subset of X. As it is shown in the proof of Theorem 5, there exists a Cauchy sequence (a_n) in A consisting of mutually different points of A. For each $n \in \mathbb{N}$, put $F_n := \{a_k : k \geq n\} \subseteq A$. Note that each $a_n \notin F_{n+1}$, since the terms of (a_n) are mutually different points of A. Further, (F_n) is a decreasing sequence of subsets of A. Put $\varepsilon_k := \frac{1}{2}d(a_k, F_{k+1})$, for each $k \in \mathbb{N}$. If there exists a $k \in \mathbb{N}$ such that $\varepsilon_k = 0$, then $a_k \in \operatorname{Cl} F_{k+1}$. We get $a_k \in (\operatorname{Cl} F_{k+1}) \setminus F_{k+1}$ and conclude that a_k is an accumulation point of F_{k+1}. Then a_k is an accumulation point of A as well, since $F_{k+1} \subseteq A$. If $\varepsilon_k > 0$, for each $k \in \mathbb{N}$, applying the previous lemma, we get a subsequence (a_{n_k}) of the sequence (a_n) such that $n_k > k$ and $d(a_{n_j}, a_{n_k}) < \varepsilon_k$, for each $j \geq k$ and each $k \in \mathbb{N}$. Let $f : F_1 \to F_1$ be a map defined by $f(a_k) = a_{n_k}$. Since $n_k > k$ and all terms of (a_n) are mutually different points, it follows $f(a_k) = a_{n_k} \neq a_k$, for each k. Hence, f does not have any fixed point. Let us show that f is a contraction with a coefficient $\kappa = \frac{1}{2}$. Take arbitrary points $a_k, a_j \in F_1, k \neq j$ and, without loss of generality, assume that $j > k$. Then $d(f(a_j), f(a_k)) = d(a_{n_j}, a_{n_k}) < \varepsilon_k = \frac{1}{2}d(a_k, F_{k+1}) \leq \frac{1}{2}d(a_k, a_j)$ holds, which proves that f is a contraction. So, f is a contraction of a subspace $F_1 \subseteq X$ which does not satisfies *BFPP*. By the assumption, each closed subspace $Y \subseteq X$ of X satisfies *BFPP*, which implies that F_1 is not a closed subset of X. Thus, $\operatorname{Cl} F_1 \neq F_1$ and there exists a point $x \in (\operatorname{Cl} F_1) \setminus F_1$. The point x is an accumulation point of F_1 and also of A, since $F_1 \subseteq A$. In both cases the derived set A' of A is non-empty and the claim is proved. □

Having in mind a way how the proof of Theorem 3 is carried out, we get the following theorem.

Theorem 7. *Let (X, d) be a metric space. Then the following claims are equivalent:*
(i) X is complete.
(ii) Each closed subspace $Y \subseteq X$ of X satisfies BFPP.
(iii) Each infinite totally bounded subset $A \subseteq X$ has an accumulation point in X.

At the end, let us go back to the subspace Y considered in Proposition 2 and use its properties to get a condition for incompleteness of metric spaces.

Corollary 8. *Let (X, d) be a metric space and let $Y = \{x_n : n \in \mathbb{N}\} \subseteq X$ be an infinite closed subspace of X consisting of mutually different points of X having a property that there exist a strictly increasing sequence (n_k) in $\mathbb{N}$ and a coefficient $\kappa \in [0, 1\rangle$ such that $n_k > k$ and $d(x_{n_k}, x_{n_j}) \leq \kappa d(x_k, x_j)$ for each $j, k \in \mathbb{N}$. Then X is not complete. Furthermore, if for each contraction $f : X \to X$ there exists a complete subspace $Z \subseteq X$ such that $f(X) \subseteq Z$, then X satisfies BFPP.*

Proof. Define $f\colon Y \to Y$ by $f(x_k) = x_{n_k}, k \in \mathbb{N}$. Then f is a contraction which does not have a fixed point. According to Theorem 3, it follows that X is incomplete. The second claim is obvious, since we can apply the Banach fixed point theorem to the restriction $f \restriction_Z$. $\square$

References

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