ACTA MATHEMATICA SPALATENSIA
Vol.1 (2021) 29–46
https://doi.org/10.32817/ams.1.1.3

Received: 30 Aug 2019
Accepted: 14 Oct 2019

On the iterated normed duals

Nikica Uglešić

Abstract

Several properties of the normed Hom-functor (dual) D and its iterations D^n are exhibited. For instance, D turns every canonical embedding (into the second dual space) to a retraction (of the third dual onto the first one) having for the right inverse the appropriate canonical embedding (of the first dual space into the third one). Some consequences to the direct-sum presentations and quotients of higher dual spaces are considered.

Keywords: normed (Banach) vectorial space, canonical embedding, normed dual Hom-functor, quotient normed space
2010 Math. Subj. Class.: Primary 54E99

1 Introduction

By studying the quotient shapes of (normed) vectorial spaces in [7] and [8], the author had realized that the iterated normed dual functors and canonical embeddings can provide some important informations about the structure of higher dual spaces. Let $\mathcal{N}_F$ ($\mathcal{B}_F \subseteq \mathcal{N}_F$) denote the category of normed (Banach) spaces over $F \in \{\mathbb{R}, \mathbb{C}\}$. In this paper we consider the normed dual functor $D \colon \mathcal{N}_F \to \mathcal{N}_F$ as well as its iterations, i.e., the contravariant and covariant functors

$$D^{2n-1} \colon \mathcal{N}_F \to \mathcal{N}_F,\ D^{2n} \colon \mathcal{N}_F \to \mathcal{N}_F,\ n \in \mathbb{N},$$

respectively (having their images in $\mathcal{B}_F \subseteq \mathcal{N}_F$).

An interesting and useful result reads as follows. For every object X of $\mathcal{N}_F$, D^{2n-1} turns the canonical embedding $j_X \colon X \to D^2(X)$ to the retraction $D^{2n-1}(j_X) \colon D^{2n+1}(X) \to D^{2n-1}(X)$ for which $D^{2n-2}(j_{D(X)})$:

$D^{2n-1}(X) \to D^{2n+1}(X)$ is an associated section ($D^0 \equiv 1_{\mathcal{N}_F}$). Thus we may think $D^n(X)$ as a closed complemented subspace of $D^{n+2}(X)$ having the annihilator $D^{n-1}(X)^0$ for a closed direct complement. As a consequence, we exhibit the following closed direct-sum presentations:

$$D^{2n+1}(X) = D_1 \dotplus A_0 \dotplus A_2 \dotplus \cdots \dotplus A_{2n-2},$$
$$D^{2n+2}(X) = D_2 \dotplus A_1 \dotplus A_3 \dotplus \cdots \dotplus A_{2n-1},$$

where (isometrically) $D_1 \cong D(X)$, $D_2 \cong D^2(X)$, $A_k \cong D^k(X)^0$, $0 \leq k \leq 2n-1$, and $D^k(X)$ is identified with its iterated canonical image in $D^{2n}(X)$, k even, or $D^{2n+1}(X)$, k odd. At the end, we aim our attention at the behavior of the iterated dual functors on a quotient space.

2 Preliminaries

We shall frequently use and apply in the sequel several general or special well known facts without referring to any source. So we remind a reader that

- the set theoretic and topological facts come from [1];
- the facts concerning functional analysis are taken from [2, 4, 5, 6];
- our category theory language follows that of [3].

Our category framework will be $\mathcal{N}_F$ — the category of all normed spaces over $F \in \{\mathbb{R}, \mathbb{C}\}$ and *all* the corresponding continuous linear function, i.e., all bounded linear operators, as well as $\mathcal{B}_F$ — its full subcategory determined by all Banach spaces. Clearly, $\mathcal{N}_F$ is a concrete category whose every object is an ordered pair $(X, \|\cdot\|)$, where X is a vectorial space over F and $\|\cdot\|$ is a norm on X (implicitly including the algebraic structure too). When there is no ambiguity, we do not stress the norm.

We recall hereby the well known dual space of a normed vectorial space over $F \in \{\mathbb{R}, \mathbb{C}\}$. For our purpose it is much more convenient to use the categorical approach as follows. There exists a contravariant structure preserving hom-functor, i.e., the contravariant Hom-functor

$$Hom_F \equiv D \colon \mathcal{N}_F \to \mathcal{N}_F,$$
$$D(X) = X^* \text{ — the normed dual space of } X,$$
$$D(f \colon X \to Y) \equiv D(f) \equiv f^* \colon Y^* \to X^*, \; D(f)(y^1) = y^1 f,$$

and $D[\mathcal{N}_F] \subseteq \mathcal{B}_F$. Furthermore, for every ordered pair $X, Y \in Ob(\mathcal{N}_F)$, the function

$$D_Y^X \colon \mathcal{N}_F(X, Y) \equiv L(X, Y) \to L(Y^*, X^*) \equiv \mathcal{B}_F(Y^*, X^*)$$

is a linear isometry ($\|D(f)\| = \|f\|$), and hence, D^X_Y belongs to $Mor(\mathcal{N}_F)$ and D is a faithful functor.

Further, there exists a covariant Hom-functor

$$
\begin{aligned}
&Hom^2_F \equiv D^2 \colon \mathcal{N}_F \to \mathcal{N}_F,\\
&D^2(X) = D(D(X)) \equiv X^{**} \text{ — the normed second dual space of } X,\\
&D^2(f\colon X \to Y) \equiv D^2(f) = D(D(f)) \equiv f^{**} \colon X^{**} \to Y^{**},\\
&\qquad\qquad\qquad\qquad\qquad\qquad D^2(f)(x^2) = x^2 D(f),
\end{aligned}
$$

and $D^2[\mathcal{N}_F] \subseteq \mathcal{B}_F$. (Caution: The notation "$D(D(f)(x^2))$" makes no sense!) Furthermore, for every ordered pair $X, Y \in Ob(\mathcal{N}_F)$, the function

$$(D^2)^X_Y \colon \mathcal{N}_F(X,Y) \equiv L(X,Y) \to L(X^{**},Y^{**}) \equiv \mathcal{B}_F(X^{**},Y^{**})$$

is a linear isometry ($\left\|D^2(f)\right\| = \|f\|$), and thus, $(D^2)^X_Y$ belongs to $Mor(\mathcal{N}_F)$ and D^2 is a faithful functor.

The most useful fact hereby is the existence of a certain natural transformation $j\colon 1_{\mathcal{N}_F} \rightsquigarrow D^2$ of the functors, where, for every X, $j_X \colon X \to D^2(X)$ is an isometric embedding (the *canonical* embedding defined by $(j_X(x))(x^1) = x^1(x)$), and $\mathrm{Cl}(j_X[X]) \subseteq D^2(X)$ is the well known (Banach) completion of X. Namely, given a pair X, Y of normed spaces, then

$$(\forall f \in \mathcal{N}_F(X,Y),\ j_Y f = D^2(f) j_X$$

holds true. Indeed, for every $x \in X$ and every $y^1 \in D^1(Y)$,

$$
\begin{aligned}
&((j_Y f)(x))(y^1) = y^1(f(x)) = j_X(x)(y^1 f) = j_X(x)(D(f)(y^1))\\
&= (j_X(x)D(f))(y^1) = (D^2(f)(j_X(x))(y^1) = ((D^2(f)j_X)(x))(y^1).
\end{aligned}
$$

Clearly, if X is a Banach space, then the canonical embedding j_X is closed. Continuing by induction, for every $n \in \mathbb{N}$, $n > 2$, there exists a Hom_F-functor D^n of $\mathcal{N}_F$ to $\mathcal{N}_F$ such that $D^n[\mathcal{N}_F] \subseteq \mathcal{B}_F$, D^n is contravariant (resp. covariant) whenever n is odd (resp. even), and for every ordered pair X, Y of normed spaces, the function $(D^n)^X_Y$ is an isometric linear morphism of the normed space $L(X,Y)$ to the Banach space $L(D^n(Y), D^n(X))$ (n odd) or $L(D^n(X), D^n(Y))$ (n even). Consequently, every $(D^n)^X_Y$ preserves null–vector morphisms, i.e., $D^n(c_\theta) = c_{\theta^n}$.

3 Some useful properties of the normed dual functor

In addition to the properties of the normed dual functor quoted in the previous section, we need the following ones too:.

Lemma 1. *(i) The functor $D\colon \mathcal{N}_F \to \mathcal{N}_F$ turns*

- *(open) epimorphisms into (closed) monomorphisms;*
- *open or closed monomorphisms and embeddings into open surjective epimorphisms;*
- *isometric isomorphisms into isometric isomorphisms.*

The functor $D^2\colon \mathcal{N}_F \to \mathcal{N}_F$ maps

- *open epimorphisms into open surjective epimorphisms;*
- *open or closed monomorphisms and embeddings into closed monomorphisms;*
- *isometries into closed isometries.*

(ii) In addition, the restriction functor $D|\mathcal{B}_F$ turns

- *surjective epimorphisms into closed monomorphisms;*
- *(isometric) monomorphisms with closed ranges into (closed) epimorphisms.*

The restriction functor $D^2|\mathcal{B}_F$ maps

- *open or surjective epimorphisms into open surjective epimorphisms;*
- *monomorphisms with closed ranges into closed monomorphisms.*

(iii) For all $X, Y \in Ob\mathcal{N}_F$, the canonical embedding

$$j_{L(X,Y)}\colon L(X,Y) \to D^2(L(X,Y))$$

factorizes trough the linear isometry

$$(D^2)^X_Y\colon L(X,Y) \to L(D^2(X), D^2(Y)),\ D^2(f)(x^2) = x^2 D(f).$$

If Y is a Banach space, then the linear isometry

$$D^X_Y\colon L(X,Y) \to L(D(Y), D(X)), D(f)(y^1) = y^1 f,$$

is closed.

Proof. (i). Assume that $f \in \mathcal{N}_F(X,Y)$ is an epimorphism. Let $y^1, y^{1\prime} \in Y^*$ such that $D(f)(y^1) = D(f)(y^{1\prime})$. It means that $y^1 f = y^{1\prime} f$, implying that $y^1 = y^{1\prime}$ because f is an epimorphism. Hence, $D(f)$ is a monomorphism of the underlying abelian groups, and consequently, it is a monomorphism of $\mathcal{B}_F \subseteq \mathcal{N}_F$. Assume, in addition, that f is open.

It suffices to prove that the range $R(D(f)) \trianglelefteq X^*$ is a closed subspace, i.e., that $\mathrm{Cl}(R(D(f))) \subseteq R(D(f))$. Namely, if it is so, then $D(f)$ is a monomorphism of a Banach space with the range that is a Banach space too. Then,

$$D(f)'\colon Y^* \to R(D(f)),\ D(f)'(y^1) = D(f)(y^1),$$

is a continuous bijection of Banach spaces, and thus, an isomorphism, implying that $D(f)$ is closed monomorphism. Let $x^1 \in \mathrm{Cl}(R(D(f)))$. Consider a sequence (x^1_n) in $R(D(f))$ such that $\lim(x^1_n) = x^1$. Since $D(f)$ is a monomorphism, it there exists a unique sequence (y^1_n) in Y^* such that, for each $n \in \mathbb{N}$,

$$D(f)(y^1_n) = y^1_n f = x^1_n.$$

Recall that, algebraically, $X = N(f) \dot{+} W$, where $W \cong R(f) = Y$, and that each fiber $f^{-1}[\{y\}]$, $y \in Y$, is the equivalence class $[x]_f = x + N(f)$, where $f(x) = y$. Thus, for every n, and every $y \in Y$,

$$y^1_n(y) = x^1_n(x) = x^1_n(w),$$

where $f(x) = y$ and $x = z + w$ is the unique presentation of $x \in X = N(f) \dot{+} W$. It implies that, for each $y \in Y$ and all $x = z + w \in X$, such that $f(x) = y$,

$$\lim(y^1_n(y)) = \lim(x^1_n(x)) = x^1(x) = x^1(w)$$

holds true. Consequently, by putting

$$y^1\colon Y \to F,\ y^1(y) = \lim(y^1_n(y)),$$

a certain function is well defined. Moreover, y^1 is linear, because it is a "copy" of the restriction $x^1|W$, and $y^1 f = x^1$ obviously holds. It remains to prove that y^1 is continuous. Let O be an open neighborhood of $0 \in F$. Since x^1 is continuous, it there exists an open neighborhood U of $\theta_X \in X$ such that $x^1[U] \subseteq O$. Then $V \equiv f[U]$ is an open neighborhood of $\theta_Y \in Y$, because f is open, and

$$y^1[V] = (y^1 f)[U] = x^1[U] \subseteq O.$$

Thus, y^1 is continuous, and hence $y^1 \in Y^*$. Since $D(f)(y^1) = y^1 f = x^1$, we have the additional statement proven.

Assume that $f\colon X \to Y$ is an open or closed monomorphism or an embedding. Then f admits the factorization

$$X \xrightarrow{f'} f[X] \overset{i}{\hookrightarrow} Y,\ f'(x) = f(x),$$

where f' is an isomorphism onto the subspace $f[X] \trianglelefteq Y$, and i is the inclusion. Given an $x^1 \in X^*$, put $y^1_{x^1} = x^1 f'^{-1} \in f[X]^*$. By the Hahn-Banach theorem, there exists an extension $y^1 \in Y^*$ of $y^1_{x^1}$, i.e., $y^1 i = y^1_{x^1}$. Then

$$\begin{aligned} D(f)(y^1) &= D(if')(y^1) = (D(f')D(i))(y^1) = D(f')(D(i)(y^1)) \\ &= D(f')(y^1 i) = D(f')(y^1_{x^1}) = y^1_{x^1} f' = x^1 f'^{-1} f' = x^1, \end{aligned}$$

implying that $D(f)\colon Y^* \to X^*$ is a surjective epimorphism. Now, by Open-mapping theorem, $D(f)$ is open. Finally, if f is an isometric isomorphism, then $D(f)$ is an isomorphism and, for every $y^1 \in D(Y)$,

$$\begin{aligned} \left\|D(f)(y^1)\right\| &= \left\|y^1 f\right\| = \sup\left\{|y^1(f(x))| \,\middle|\, x \in X, \|x\| = 1\right\} \\ &= \sup\left\{|y^1(y)| \,\middle|\, y \in Y, \|y\| = \|f(x)\| = \|x\| = 1\right\} = \left\|y^1\right\|. \end{aligned}$$

Hence, $D(f)$ is an isometry as well. The statements concerning D^2 follow by $D^2(f) = D(D(f))$ and $D^2(f)j_X = j_Y f$, where j_X and j_Y are the (isometric) canonical embeddings.
(ii). Assume that $f \in \mathcal{B}_F(X,Y)$ is a surjective epimorphism. By Open-mapping theorem, f is open as well. Then, by (i), $D(f)$ is a closed monomorphism.
Assume that $f \in \mathcal{B}_F(X,Y)$ is a monomorphism having the range $R(f)$ closed in Y. Then, as previously,

$$f'\colon X \to R(f),\ f'(x) = f(x),$$

is a continuous bijection of Banach spaces, and thus, an isomorphism. It follows that f is a closed monomorphism. Then, by (i), $D(f)$ is an open surjective epimorphism. If, in addition, f is an isometry, then f preserves Cauchy sequences, and one readily verifies that $D(f)$ maps the sets closed in $D(Y)$ into sets closed in $D(X)$. The statements concerning $D^2|\mathcal{B}_F$ follow by those concerning $D|\mathcal{B}_F$.
(iii). Consider the range

$$R((D^2)^X_Y) \equiv (D^2)^X_Y[L((X,Y))] \trianglelefteq L(D^2(X), D^2(Y))$$

and the function

$$u\colon R((D^2)^X_Y) \to D^2(L(X,Y))$$

well defined by $u(D^2(f)) = f^2_f$ such that, for each $f^1 \in D(L(X,Y))$, $f^2_f(f^1) = f^1(f)$. One readily sees that u is linear and continuous, and that $j_{L(X,Y)} = u(D^2)^X_Y$.

Finally, let Y be a Banach space, and let $C \subseteq L(X,Y)$ be a closed set. Let (g_n) be sequence in $D[C]$ that converges in $L(D(Y), D(X))$, i.e., there exists $\lim(g_n) \equiv g \in L(D(Y), D(X))$. Since D_Y^X is a linear isometry, it is a monomorphism, and there exists a unique Cauchy sequence (f_n) in C such that, for each $n \in \mathbb{N}$, $D(f_n) = g_n$. Notice that $L(X,Y)$ is a Banach space because such is Y, and thus, there exists $\lim(f_n) \equiv f \in L(X,Y)$. Since $C \subseteq L(X,Y)$ is closed, it follows that $f \in C$. Then $D(f) \in D[C]$, and the continuity implies that $D(f) = g$, which completes the proof. □

Lemma 2. *For every normed space X, the canonical embedding*

$$j_{D(X)} \colon D(X) \to D^2(D(X)) = D^3(X)$$

is a section of $\mathcal{B}_F$ having $D(j_X) \colon D^3(X) \to D(X)$ for the corresponding retraction, i.e., $D(j_X)j_{D(X)} = 1_{D(X)}$, and $D(X) \equiv R(j_{D(X)})$ admits a closed direct complement in $D^3(X)$.

Proof. Given any $X \in Ob(\mathcal{N}_F)$, we have to prove that $D(j_X)j_{D(X)} = 1_{D(X)}$. Recall that, in general, $j_Y \colon Y \to D^2(Y)$ is defined by $j_Y(y^0) = y^2_{y^0}$, $y^0 \in Y$, such that, for every $y^1 \in D(Y)$, $y^2_{y^0}(y^1) = y^1(y^0)$. Further,

$$D(j_Y) \colon D(D^2(Y)) = D^3(Y) \to D(Y)$$

is determined by $D(j_Y)(y^3) = y^3 j_Y$, $y^3 \in D^3(Y)$. In the same way, the canonical embedding

$$j_{D(Y)} \colon D(Y) \to D^2(D(Y)) = D^3(Y) = D(D^2(Y))$$

is determined by $j_{D(Y}(y^1) = y^3_{y^1}$, $y^1 \in D(Y)$, such that, for every $y^2 \in D^2(Y)$, $y^3_{y^1}(y^2) = y^2(y^1)$. Thus, for $Y = D(X)$ and every $x^1 \in D(X)$,

$$(D(j_X)j_{D(X)})(x^1) = D(j_X)(j_{D(X)}(x^1)) = D(j_X)(x^3_{x^1}) = x^3_{x^1} j_X.$$

Since, in addition, for every $x^0 \in X$, we have

$$(x^3_{x^1} j_X)(x^0) = x^3_{x^1}(j_X(x^0)) = x^3_{x^1}(x^2_{x^0}) = x^2_{x^0}(x^1) = x^1(x^0),$$

it follows that, for every $x^1 \in D(X)$, $x^3_{x^1} j_X = x^1$ holds, and therefore, $D(j_X)j_{D(X)} = 1_{D(X)}$. This proves the first claim. Further, notice that

$$p_{D(X)} \equiv j_{D(X)} D(j_X) \colon D^3(X) \to D^3(X)$$

is a (continuous linear) projection ($p^2_{D(X)} = p_{D(X)}$, of norm 1) onto $R(j_{D(X)})$. Therefore, the Banach space $D(X)$, identified with $j_{D(X)}[D(X)] \equiv R(j_{D(X)})$, admits a closed direct complement in $D^3(X)$. □

It is well known that there are Banach spaces (for instance c and c_0) that are not *isometrically isomorphic* to any of the dual normed spaces. We now prove that they are not even isomorphic in general.

Theorem 3. *(i) If a normed space X is isomorphic to a dual space of a normed space, then X is a Banach space, the canonical embedding $j_X \colon X \to D^2(X)$ is a section of $\mathcal{B}_F$ and $X \equiv R(j_X)$ admits a closed direct complement in $D^2(X)$.*
(ii) The (codomain restriction) functor $D \colon \mathcal{N}_F \to \mathcal{B}_F$ is not surjective onto $Ob(\mathcal{B}_F)$, that is, there exists $X \in Ob(\mathcal{B}_F)$ such that $X \ncong D(Y)$ for each $Y \in Ob(\mathcal{N}_F)$.

Proof. (i). Let X be a normed space such that $X \cong D^n(Y)$ for some $Y \in Ob(\mathcal{N}_F)$ and $n \in \mathbb{N}$. Then, clearly, X is a Banach space. Since $D^n(Y) = D(D^{n-1}(Y))$, one may assume that $n = 1$. Let $f \colon X \to D(Y)$ be an isomorphism of $\mathcal{B}_F \subseteq \mathcal{N}_F$. Then $D^2(f) \colon D^2(X) \to D^3(Y)$ is an isomorphism of $\mathcal{B}_F$ and the diagram

$$\begin{array}{ccc} X & \stackrel{f}{\to} & D(Y) \\ j_X \downarrow & & \downarrow j_{D(Y)} \\ D^2(X) & \underset{D^2(f)}{\to} & D^3(Y) \end{array}$$

in $\mathcal{B}_F$ commutes. By Lemma 2, $D(j_Y) j_{D(Y)} = 1_{D(Y)}$ holds true. Put

$$r_X \colon D^2(X) \to X, \ r_X = f^{-1} D(j_Y) D^2(f).$$

Then one readily verifies that $r_X j_X = 1_X$ and hence $j_X \colon X \to D^2(X)$ is a section of $\mathcal{B}_F$. Further, notice that the morphism

$$p_X \equiv j_X r_X : D^2(X) \to D^2(X)$$

is a continuous linear projection ($p_X^2 = p_X$) onto $R(j_X)$. Therefore, the Banach space X, identified with $j_X[X] \equiv R(j_X)$, admits a closed direct complement in $D^2(X)$. (Notice that $\|p_X\| = \|r_X\| = 1$ regardless to $\|f\|$.)
(ii). Assume to the contrary, i.e., that every Banach space is isomorphic to the dual space of a normed space. Then, since the dual of a space equals to the dual of its Banach completion, every Banach space Z is isomorphic to the dual $D(W)$ of a Banach space W. (Further, by iteration, every Banach space would be isomorphic to the second dual of a Banach space, and so on.) Thus by our assumption and (i), for every Banach space Z, $j_Z \colon Z \to D^2(Z)$ is a section of $\mathcal{B}_F$ and $Z \cong R(j_Z)$ admits a closed direct complement in $D^2(Z)$. Now, let X be a non-bidual-like Banach space, i.e., $D^2(X) \ncong X$. Then $X \cong R(j_X) \subsetneq D^2(X)$. Now assume

that there is a closed subspace $Z \trianglelefteq D^2(X)$ such that $R(j_X) \trianglelefteq Z$. Choose an arbitrary such Z and denote by $j\colon X \hookrightarrow Z$ the codomain restriction of j_X. By Lemma 1 (i), we may assume that $D^2(X) \cong D^2(R(j_X)) \trianglelefteq D^2(Z)$ as well. According to (i), let

$$r_Z\colon D^2(Z) \to Z,\ r_Z j_Z = 1_Z$$

be a retraction of $\mathcal{B}_F$ corresponding to the canonical embedding j_Z. Then the domain restriction

$$r \equiv r_Z|D^2(X)\colon D^2(X) \to Z,\ ri = 1_Z,$$

is a (continuous linear) retraction of $D^2(X)$ onto the subspace Z, implying that

$$p \equiv ir\colon D^2(X) \to D^2(X)$$

is a continuous linear projection ($p^2 = p$) along $N(p) = N(r)$ onto $R(p) = R(r) = Z$. This implies that every such Z admits a closed direct complement in $D^2(X)$. Finally, in order to get a contradiction, an appropriate pair X, Z of such concrete Banach spaces is needed. Let $X = c_0$ (the subspace of l_∞ consisting of all null-convergent sequences in F). Recall that c_0 is not bidual-like because of $D^2(c_0) \cong l_\infty \ncong c_0$. Namely, there are (isometric) isomorphisms $D(c_0) \cong l_1$ and $D(l_1) \cong l_\infty$ (see [5, p. 86, Zadaci 6. and p. 76, Teorem 2.3.8]). However, it is well known that there is a closed subspace $Z \trianglelefteq l_\infty$,

$$c_0 \equiv R(j_{c_0}) \trianglelefteq Z \trianglelefteq l_\infty \cong D^2(c_0),$$

which does not admit any closed direct complement in l_∞ — a contradiction. This completes the proof. □

Remark 4. *(i) Though, by Theorem 3, there are Banach spaces that are not isomorphic to any of the dual spaces, we do not know whether every Banach space is a retract of its second dual space (the converse of Theorem 3 (i) (?)). (ii) Recall that $D(F^n) \cong F^n$, $n \in \mathbb{N}$, and that, for all $1 < p, q < \infty$ such that $p^{-1} + q^{-1} = 1$, we have*

$$D(F_0^{\mathbb{N}}, \|\cdot\|_p) = D(Cl_{l_p}(F_0^{\mathbb{N}}, \|\cdot\|_p)) = D(l_p) \cong l_q,$$

where $F_0^{\mathbb{N}}$ is the countable direct sum of F's, i.e.,

$$F_0^{\mathbb{N}} = \{x\colon \mathbb{N} \to F \mid x(n) = 0 \textit{ for almost all } n\}.$$

Further, since $D(l_1) \cong l_\infty$ and $Cl_{l_\infty}(F_0^{\mathbb{N}}, \|\cdot\|_\infty)) = c_0$, we have

$$D(F_0^{\mathbb{N}}, \|\cdot\|_\infty) = D(c_0) \cong l_1$$

(see Lemma 4.1 (i) of [8]). Thus the following question occurs:

Question. ***Does the functor D raise an uncountable-infinite algebraic dimension?***

4 The direct-sum presentations of iterated dual spaces

Lemma 2 motivates the following consideration. Given a normed space X and a $k \in \{0\} \cup \mathbb{N}$, let us denote by $j_{k,X} \equiv j_{D^k(X)} \colon D^k(X) \to D^{k+2}(X)$ the canonical embedding ($D^0 = 1_{\mathcal{N}_F}$). Then the class $\{j_{k,X} \mid X \in Ob(\mathcal{N}_F)\}$ determines a natural transformation $j_k \colon D^k \rightsquigarrow D^{k+2}$ of the functors. When there is no ambiguity, i.e., when a normed space X is fixed, we simplify the notation $j_{k,X}$ to j_k. Notice that, for a given $X \in Ob(\mathcal{N}_F)$, the following morphisms of $\mathcal{B}_F \subseteq \mathcal{N}_F$ occur:

$$D(X) \xrightarrow{j_1} D^3(X) \xrightarrow{D(j_0)} D(X),$$

$$D^2(X) \underset{D^2(j_0)}{\overset{j_2}{\rightrightarrows}} D^4(X) \xrightarrow{D(j_1)} D^2(X),$$

$$D^3(X) \underset{D^2(j_1)}{\overset{j_3}{\rightrightarrows}} D^5(X) \underset{D(j_2)}{\overset{D^3(j_0)}{\rightrightarrows}} D^3(X)$$

and, generally, for every $k \in \{0\} \cup \mathbb{N}$ and each l, $0 \leq l \leq k$,

$$D^{2k+1}(X) \xrightarrow{D^{2k-2l}(j_{2l+1})} D^{2k+3}(X) \xrightarrow{D^{2l+1-2l}(j_{2l})} D^{2k+1}(X),$$

$$D^{2k+2}(X) \xrightarrow{D^{2k-2l}(j_{2l+2})} D^{2k+4}(X) \xrightarrow{D^{2k+1-2l}(j_{2l+1})} D^{2k+2}(X).$$

Let, for each k, $S_{2k+1}(X)$ be the set of all $D^{2k-2l}(j_{2l+1}) \in L(D^{2k+1}(X), D^{2k+3}(X))$, and let $R_{2k+1}(X)$ be the set of all $D^{2k+1-2l}(j_{2l}) \in L(D^{2k+3}(X), D^{2k+1}(X))$, $0 \leq l \leq k$. Similarly, let $S_{2k+2}(X)$ be the set of all $D^{2k-2l}(j_{2l+2}) \in L(D^{2k+2}(X), D^{2k+4}(X))$, and let $R_{2k+2}(X)$ be the set of all $D^{2k+1-2l}(j_{2l+1}) \in L(D^{2k+4}(X), D^{2k+2}(X))$, $0 \leq l \leq k$. Hence, for each $n \in \mathbb{N}$, the sets $S_n(X)$ and $R_n(X)$ are well defined. By Lemma 1, since all j_k are isometries, all the morphisms belonging to $S_n \cup R_n$ have norm 1.

Theorem 5. *(i) For every normed space X and every $n \in \mathbb{N}$, each $s \in S_n(X)$ is a section of $\mathcal{B}_F$ having an $r_s \in R_n(X)$ for the corresponding retraction, and conversely, each $r \in R_n(X)$ is a retraction of $\mathcal{B}_F$ having an $s_r \in R_n(X)$ for the corresponding section. More precisely, for each $k \in \mathbb{N}$ and every $l \in \{0, \ldots, k\}$,*

$$D^{2k+1-2l}(j_{2l})D^{2k-2l}(j_{2l+1}) = 1_{D^{2k+1}(X)} \qquad \textit{and}$$
$$D^{2k+1-2l}(j_{2l+1})D^{2k-2l}(j_{2l+2}) = 1_{D^{2k+2}(X)}$$

hold true.

(ii) In general,

$(\forall n \geq 3)(\exists s \in S_n(X))(\exists r \in R_n(X))$ *rs is not an epimorphism (especially, $rs \neq 1_{D^n(X)}$.).*

Proof. Let X be a normed space. Since, for every $k \in \{0\} \cup \mathbb{N}$, the canonical morphism $j_k \colon D^k(X) \to D^{k+2}(X)$ ($D^0 = 1_{\mathcal{N}_F}$) is an isometric embedding, Lemma 1 implies that $D(j_k) \colon D^{k+3}(X) \to D^{k+1}(X)$ is an open surjective epimorphism.

(i). Let $n = 1$. By Lemma 2, $D(j_0)j_1 = 1_{D(X)}$, i.e., $j_1[D(X)]$ is a retract of $D^3(X)$ with the retraction $D(j_0)$ and the corresponding section j_1. Let $n \geq 2$. Since D is a contravariant functor, it follows that

$$D(j_1)D^2(j_0) = D(D(j_0)j_1) = D(1_{D(X)}) = 1_{D^2(X)}.$$

Therefore, $D^2(j_0)[D^2(X)]$ is a retract of $D^4(X)$ with the retraction $D(j_1)$ having $D^2(j_0)$ for the corresponding section. In general, by considering $D^n(X)$ as $D(D^{n-1}(X))$, i.e., the canonical embedding $j_n \colon D^n(X) \to D^{n+2}(X)$ as "$j_1 \colon D(D^{n-1}(X) \to D^3(D^{n+1}(X))$", and $D(j_{n-1}) \colon D^{n+2}(X) \to D^n(X)$ as "$D(j_0) \colon D^3(D^{n-1}(X)) \to D(D^{n-1}(X))$", one proves (by mimicking the appropriate part of the proof of Lemma 2) that

$$D(j_{n-1})j_n = 1_{D^n(X)}$$

holds true. Thus, for every $n \in \mathbb{N}$, $j_n[D^n(X)]$ is a retract of $D^{n+2}(X)$ with the retraction $D(j_{n-1})$ having j_n for the corresponding section. Further, since D^2 is a covariant functor, one readily verifies that, for every $k \in \mathbb{N}$ and every $l \in \{0, \dots, k\}$,

$$\begin{aligned} D^{2k+1-2l}(j_{2l})D^{2k-2l}(j_{2l+1}) &= D^{2k-2l}(D(j_{2l})j_{2l+1}) \\ &= D^{2k-2l}(1_{D^{2l+1}(X)}) = 1_{D^{2k+1}(X)}, \\ D^{2k+1-2l}(j_{2l+1})D^{2k-2l}(j_{2l+2}) &= D^{2k-2l}(D(j_{2l+1})j_{2l+2}) \\ &= D^{2k-2l}(1_{D^{2l+2}(X)}) = 1_{D^{2k+2}(X)}. \end{aligned}$$

This shows that all $D^{2k-2l}(j_{2l+1})$ and $D^{2k-2l}(j_{2l+2})$ are sections having $D^{2l+1-2l}(j_{2l})$ and $D^{2l+1-2l}(j_{2l+1})$ for the corresponding retractions, respectively, and vice versa. Notice that every $s \in S_n(X)$ (resp. $r \in R_n(X)$) can be explicitly written as $D^m(j)$ (resp. $D^{m'}(j')$) for some appropriate m and j (resp. m' and j'). Therefore, statement (i) holds true.

(ii). Let firstly $n = 3$. We are to show that, in general, for the section $s = j_3 \colon D^3(X) \to D^5(X)$ and the retraction $r = D^3(j_0) \colon D^5(X) \to D^3(X)$, the composite

$$rs = D^3(j_0)j_3 : D^3(X) \to D^3(X)$$

may be not an epimorphism. Consider the following diagram

$$\begin{array}{ccc} D^3(X) & \overset{D(j_0)}{\rightarrow} & D(X) \\ j_3 \downarrow & & \downarrow j_1 \\ D^5(X) & \underset{D^3(j_0)}{\rightarrow} & D^3(X) \end{array}$$

in $\mathcal{B}_F \subseteq \mathcal{N}_F$. Since $D^3 = D^2 D$, $D^5 = D^2 D^3$, $j_1 = j_{D(X)}$, $j_3 = j_{D^3(X)}$ and $j\colon 1_{\mathcal{N}_F} \rightsquigarrow D^2$ is a natural transformation of the functors, we conclude that the diagram commutes, i.e., $D^3(j_0)j_3 = j_1 D(j_0)$. Notice that, in general, the canonical embedding j_1 is not an epimorphism., and the conclusion follows. If $n = 4$, then one similarly proves that, for instance, $D^3(j_1)j_4$ is not an epimorphism. Generally, if an $rs : D^n(X) \to D^n(X)$ factorizes through a j_{n-2k}, $1 \leq k \leq n-2$, then, generally, it is not an epimorphism. Thus, statement (ii) follows. $\square$

Recall that, for a subset $S \subseteq X \in Ob(\mathcal{N}_F)$, the *annihilator* of S (with respect to X) is

$$S^0_X \equiv S^0 = \{x^1 \in D(X) \mid R(x^1|S) = \{0\}\} \subseteq D(X),$$

and that S^0 is a closed subspace of $D(X)$. The next theorem is an immediate consequence of Theorem 5 and several known facts from [2, Chap.6, Sect.6].

Theorem 6. *(i) For every normed space X and each $n \in \mathbb{N}$, the range $R(j_n)$ and the annihilator $R(j_{n-1})^0$ (of $R(j_{n-1})$ with respect to $D^{n+2}(X)$) are closed complementary subspaces of $D^{n+2}(X)$, i.e., by identifying $D^{n-1}(X)$ with $R(j_{n-1})$ and $D^n(X)$ with $R(j_n)$, the closed direct-sum presentation*

$$D^{n+2}(X) = D^n(X) \dotplus D^{n-1}(X)^0$$

holds true. Consequently, by iteration,

$$D^{2n+1}(X) = D_1 \dotplus A_0 \dotplus A_2 \dotplus \cdots \dotplus A_{2n-2},$$
$$D^{2n+2}(X) = D_2 \dotplus A_1 \dotplus A_3 \dotplus \cdots \dotplus A_{2n-1},$$

where (isometrically) $D_1 \cong D(X)$, $D_2 \cong D^2(X)$, $A_k \cong D^k(X)^0$, $0 \leq k \leq 2n-1$, and $D^k(X)$ is identified with its iterated canonical image in $D^{2n}(X)$, k even, or $D^{2n+1}(X)$, k odd.
(ii) If X is a normed space admitting a retraction $r_0\colon D^2(X) \to Cl(j_0[X]) \equiv \bar{X}$ (in $\mathcal{B}_F$), then

$$D^2(X) \cong \bar{X} \dotplus N(r_0)$$

is a closed direct-sum presentation of $D^2(X)$. If, in addition, X is a Banach space, then $D^2(X) \cong X \dotplus N(r_0)$ is a closed direct-sum presentation, where X is identified with $R(j_0)$.

Proof. (i). Notice that, for a given X and each $n \in \mathbb{N}$,

$$p_{n+2} \equiv j_n D(j_{n-1})\colon D^{n+2}(X) \to D^{n+2}(X)$$

is a continuous linear projection. Indeed,

$$p_{n+2}^2 = (j_n D(j_{n-1}))(j_n D(j_{n-1})) = j_n 1_{D^n(X)} D(j_{n-1}) = j_n D(j_{n-1}) = p_{n+2}.$$

Since $D(j_{n-1})$ is a surjective epimorphism. and j_n is an isometric embedding, it follows that

$$R(p_{n+2}) = R(j_n) \cong D^n(X).$$

Further,

$$\begin{aligned} N(p_{n+2}) &= N(D(j_{n-1})) = \{x^{n+2} \in D^{n+2}(X) \mid x^{n+2} j_{n-1} = c_0^{n-1}\} \\ &= (j_{n-1}[D^{n-1}(X)])^0 \equiv R(j_{n-1})^0. \end{aligned}$$

Now the conclusion follows by induction and the well known general facts. (Observe that, for instance,

$$p'_{n+2} \equiv D^2(j_{n-2}) D(j_{n-1})\colon D^{n+2}(X) \to D^{n+2}(X),\ n > 1,$$

is also a continuous linear projection yielding another closed direct-sum presentation of $D^{n+2}(X)$.)
(ii). If X admits a retraction $r_0\colon D^2(X) \to Cl(j_0[X]) \equiv \bar{X}$, then

$$p_2 \equiv j_0 r_0\colon D^2(X) \to D^2(X)$$

is a continuous linear projection. Since the both $R(p_2) = Cl(R(j_0)) \equiv \bar{X}$ and $N(p_2) = N(r_0)$ are closed in $D^2(X)$, we infer that the stated closed direct-sum presentation follows. If such an X is a Banach space, one may identify $X \equiv \bar{X} \subseteq D^2(X)$, and the conclusion follows. $\square$

Example 7. *Recall that $D(c) \cong l_1$ ($\cong D(c_0)$) and $D(l_1) \cong l_\infty$. Then, by Theorem 6 (i),*

$$D(l_\infty) \cong D^3(c) \cong D(c) \dotplus c^0 \cong A \dotplus c^0,$$

where $A \trianglelefteq l_\infty$ and $A \cong l_1$. This also shows that the annihilator does not preserve separability of a subspace.

5 The iterated dual functors and quotients

By the mentioned identifications, Theorem 6 shows that $D^{n+2}(X)/D^n(X)$ is (isometrically) isomorphic to $D^{n-1}(X)^0$. Further, it is well known that $D(X)/Z^0$, $Z \trianglelefteq X$, is isometrically isomorphic to $D(Z)$, and that, for Banach spaces, $D(X/Z)$ is isometrically isomorphic to Z^0. These facts and Theorem 5 aim our attention at the behavior of the iterated dual functors on a quotient space (see also [2, Chapter 6., Sections 5. and 6.]).

Lemma 8. *Let Z be a closed subspace of a normed space X, and denote by $i\colon Z \hookrightarrow X$ the inclusion, and by $q\colon X \to X/Z$ the quotient morphism. Then, for every $n \in \mathbb{N}$, the short sequences*

$$D^{2n-1}(Z) \overset{D^{2n-1}(i)}{\longleftarrow} D^{2n-1}(X) \overset{D^{2n-1}(q)}{\longleftarrow} D^{2n-1}(X/Z) \quad \textit{and}$$

$$D^{2n}(Z) \overset{D^{2n}(i)}{\longrightarrow} D^{2n}(X) \overset{D^{2n}(q)}{\longrightarrow} D^{2n}(X/Z)$$

in $\mathcal{B}_F$ are exact.

Proof. Clearly, the short sequence

$$Z \overset{i}{\hookrightarrow} X \overset{q}{\to} X/Z$$

in $\mathcal{N}_F$ is exact, i.e., $R(i) = N(q)$. Since D is a contravariant functor and the function $D^Z_{X/Z} : L(X, X/Z) \to L(D(X/Z), D(X))$ is linear, it follows that

$$D(i)D(q) = D(qi) = D(c_\theta) = c_{\theta^1},$$

i.e., $R(D(q)) \subseteq N(D(i))$. We are to prove that the converse $N(D(i)) \subseteq R(D(q))$ holds as well. Let $x^1 \in N(D(i)) \subseteq X^*$, i.e., $D(i)(x^1) = x^1 i = c_0$ which implies that $x^1[Z] = \{0\}$, i.e., $x^1 \in Z^0$. By the universal property of the quotient morphism q, there exists a continuous linear function

$$w_{x^1} : X/Z \to F,\ w_{x^1}([x]) \equiv w_{x^1} q(x) = x^1(x).$$

Then, clearly, $w_{x^1} \in (X/Z)^*$ and, moreover,

$$D(q)(w_{x^1}) = w_{x^1} q = x^1,$$

implying that $x^1 \in R(D(q))$, which proves the converse. Hence, the short sequence

$$D(Z) \overset{D(i)}{\leftarrow} D(X) \overset{D(q)}{\leftarrow} D(X/Z)$$

in $\mathcal{B}_F$ is exact. Further, by Lemma 1, $D(i)\colon D(X) \to D(Z)$ is an epimorphism., and thus, the range $R(D(i)) = D(Z)$ is (trivially) closed in $D(Z)$. Then, by Proposition 6.5.13. of [2], the short sequence

$$D^2(Z) \overset{D^2(i)}{\to} D^2(X) \overset{D^2(q)}{\to} D^2(X/Z)$$

in $\mathcal{B}_F$ is exact. Now, in general, by Lemma 1, $D^{2n}(q)$ and $D^{2n+1}(i)$ are epimorphisms, i.e., $R(D^{2n}(q)) = D^{2n}(X/Z)$ and $R(D^{2n+1}(i)) = D^{2n+1}(Z)$. (It suffices that $R(D^{2n}(q)$ is closed in $D^{2n}(X/Z)$ and that $R(D^{2n+1}(i))$ is closed in $D^{2n+1}(Z)$, which follows by Proposition 6.5.12. of [2].) Then the final conclusion follows by Proposition 6.5.13. of [2]. □

We can now state the following general facts concerning the iterated dual functors and quotients.

Theorem 9. *Let Z be a closed subspace of a normed space X, and denote by $i\colon Z \hookrightarrow X$ the inclusion, and by $q\colon X \to X/Z$ the quotient morphism. Then, for each $n \in \mathbb{N}$,*
(i) the functor D^{2n-1} permits cancellation on the quotient objects, i.e.,

$$D^{2n-1}(X)/D^{2n-1}(X/Z)) \cong D^{2n-1}(Z),$$

where $D^{2n-1}(X/Z)$ is identified with $R(D^{2n-1}(q))$ in $D^{2n-1}(X)$;
(ii) the functor D^{2n} "preserves" the quotient of objects, i.e.,

$$D^{2n}(X/Z) \cong D^{2n}(X)/D^{2n}(Z),$$

where $D^{2n}(Z)$ is identified with $R(D^{2n}(i))$ in $D^{2n}(X)$.
(iii) $D(X/Z) \cong Z^0$, $D^{2n+1}(X/Z) \cong D^{2n}(Z)^0$, $D^{2n}(Z) \cong D^{2n-1}(X/Z)^0$, where $D^{2n}(Z)$ (resp. $D^{2n-1}(X/Z)$) is identified with $R(D^{2n}(i))$ (resp. $R(D^{2n-1}(q))$ in $D^{2n}(X)$ (resp. $D^{2n-1}(X)$), and all the isomorphisms are isometric.

Proof. Let $i\colon Z \hookrightarrow X$ be the inclusion and let $q\colon X \to X/Z$ be the quotient function that is an open surjective epimorphism. Consider the exact sequence

$$Z \overset{i}{\hookrightarrow} X \overset{q}{\to} X/Z, \qquad R(i) \equiv i[Z] = N(q),$$

in $\mathcal{N}_F$. By Lemma 8, for each $n \in \mathbb{N}$, the sequences

$$D^{2n-1}(Z) \overset{D^{2n-1}(i)}{\longleftarrow} D^{2n-1}(X) \overset{D^{2n-1}(q)}{\longleftarrow} D^{2n-1}(X/Z), \quad \text{and}$$
$$D^{2n}(Z)) \overset{D^{2n}(i)}{\longrightarrow} D^{2n}(X) \overset{D^{2n}(q)}{\longrightarrow} D^{2n}(X/Z),$$

in $\mathcal{B}_F$ are exact, i.e.,

$$R(D^{2n-1}(q)) \equiv D^{2n-1}(q)[D^{2n-1}(X/Z)]) = N(D^{2n-1}(i)),$$
$$R(D^{2n}(i)) \equiv D^{2n}(i)[D^{2n}(Z)] = N(D^{2n}(q))$$

Observe that, by Lemma 1, the morphisms $D^{2n-1}(q)$ and $D^{2n}(i)$ are closed monomorphisms, while $D^{2n-1}(i)$ and $D^{2n}(q)$ are open surjective epimorphisms. By the universal property of a quotient in $\mathcal{N}_F$, there exists a unique continuous linear (canonical) factorization of $D^{2n-1}(i)$ trough the quotient morphism

$$q_{2n-1} \colon D^{2n-1}(X) \to D^{2n-1}(X)/N(D^{2n-1}(i)),$$
$$q_{2n-1}(x^{2n-1}) = [x^{2n-1}],$$

such that $D^{2n-1}(i) = h_{2n-1}q_{2n-1} \in Mor(\mathcal{B}_F)$, where

$$h_{2n-1} \colon D^{2n-1}(X)/N(D^{2n-1}(i)) \to D^{2n-1}(Z),$$
$$h_{2n-1}([x^{2n-1}]) = D^{2n-1}(i)(x^{2n-1}) = x^{2n-1}D^{2n-2}(i).$$

By the same reason, there exists the canonical factorization $D^{2n}(q) = h_{2n}q_{2n}$, where

$$q_{2n} \colon D^{2n}(X) \to D^{2n}(X)/N(D^{2n}(q)),\ q_{2n}(x^{2n}) = [x^{2n}], \quad \text{and}$$
$$h_{2n} \colon D^{2n}(X)/N(D^{2n}(i)) \to D^{2n}(X/Z),$$
$$h_{2n}([x^{2n}]) = D^{2n}(q)(x^{2n}) = x^{2n}D^{2n-1}(q).$$

Since $D^{2n-1}(i)$ and $D^{2n}(q)$ are open surjective epimorphisms, so are h_{2n-1} and h_{2n} (Open-mapping theorem). Further, the above exactness, i.e.,

$$R(D^{2n-1}(q)) = N(D^{2n-1}(i)),\ R(D^{2n}(i)) = N(D^{2n}(q)$$

imply, respectively, that

$$D^{2n-1}(X)/R(D^{2n-1}(q)) = D^{2n-1}(X)/N(D^{2n-1}(i)),$$
$$D^{2n}(X)/R(D^{2n}(i)) = D^{2n}(X)/N(D^{2n}(q)).$$

Therefore, h_{2n-1} and h_{2n} are bijections. Finally, by the Banach inverse-mapping theorem, h_{2n-1} and h_{2n} are isomorphisms of $\mathcal{B}_F$. Since $D^{2n-1}(q)$ and $D^{2n}(i)$ are closed monomorphisms, one may identify $D^{2n-1}(X/Z)$ with $D^{2n-1}(q)[D^{2n-1}(X/Z)]$ in $D^{2n-1}(X)$ as well as $D^{2n}(Z)$ with $D^{2n}(i)[D^{2n}(Z)]$ in $D^{2n}(X)$. Consequently,

$$D^{2n-1}(X)/D^{2n-1}(X/Z)) \cong D^{2n-1}(Z) \quad \text{and}$$
$$D^{2n}(X)/D^{2n}(Z)) \cong D^{2n}(X/Z).$$

In this way we have proven the isomorphism relations in statements (i) and (ii).
In order to prove statement (iii) (see also Remark 10 below), let us again consider the starting exact sequence in $\mathcal{N}_F$, $R(i) = Z = N(q)$. Notice that

$$Z^0 = \{x^1 \in X^* \mid R(x^1|Z) = \{0\}\} = \{x^1 \mid x^1 i = c_0\}$$
$$= N(D(i)) = R(D(q)),$$

implying that

$$R(D(q)) \equiv D(q)[(X/Z)^*] = Z^0 \text{ in } X^*.$$

Since, by Lemma 1, $D(q)$ is a closed monomorphism, one may identify $(X/Z)^*$ with Z^0 in X^*, and then (the well known) $D(X/Z) \cong Z^0$ isometrically holds. Let $n = 1$, and let us consider the exact sequences

$$D(Z) \overset{D(i)}{\leftarrow} D(X) \overset{D(q)}{\leftarrow} D(X/Z), \quad \text{and}$$
$$D^2(Z)) \overset{D^2(i)}{\rightarrow} D^2(X) \overset{D^2(q)}{\rightarrow} D^2(X/Z)$$

from above. By arguing as previously, we may identify $D(X/Z)$ with $R(D(q))$ and $D^2(Z)$ with $R(D^2(i))$, and then conclude that $D^3(X/Z) = D(D^2(X/Z)) \cong D^2(Z)^0$ and $D^2(Z) = D(D(Z)) \cong D(X/Z)^0$ isometrically. In the general case of an $n \in \mathbb{N}$, the proof goes in the quite similar way. □

Remark 10. *A well-known isometry $X^*/Y^0 \cong Y^*$ for every subspace Y of X ([5, Section 8. 12, Propozicija 17, p. 444]) is closely related to the case $n = 1$ of Theorem 9. One can show that statements (i) and (ii) of Theorem 9 with that fact imply (iii), and conversely, statement (iii) with that fact implies (i) and (ii) of Theorem 9. Further, notice that, by Theorem 9 (iii), $(Z_X^0)_{D(X)}^0 \cong D^2(Z)$ isometrically, and especially, $(X_X^0)_{D(X)}^0 = D^2(X)$.*

References

[1] J. Dugundji, *Topology*, Allyn and Bacon, Inc. Boston, 1978.

[2] J. M. Erdman, *Functional Analysis and Operator Algebras — An Introduction*, Version October 4 2015., Portland State University (licensed PDF)

[3] H. Herlich and G. E. Strecker, *Category Theory, An Introduction*, Allyn and Bacon Inc., Boston, 1973.

[4] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New York, 1989.

[5] S. Kurepa, *Funkcionalna analiza: elementi teorije operatora*, Školska knjiga, Zagreb, 1990.

[6] W. Rudin, *Functional Analysis, Second Edition*, McGraw-Hill, Inc., New York, 1991.

[7] N. Uglešić, *On the quotient shape of vectorial spaces*, Rad HAZU – Matematičke znanosti, **21** = **532** (2017), 179–203.

[8] N. Uglešić, *The quotient shapes of* l_p *and* L_p *spaces*, Rad HAZU – Matematičke znanosti, **27** = **536** (2018), 149–174.

Nikica Uglešić
Sv. Ante 9, 23287 Veli Råt, Hrvatska (Croatia)
E-mail address: nuglesic@unizd.hr.