ACTA MATHEMATICA SPALATENSIA
Vol.1 (2021) 19–27
https://doi.org/10.32817/ams.1.1.2

Received: 07 Apr 2019
Accepted: 12 Jun 2019

Rational Diophantine sextuples containing two regular quadruples and one regular quintuple

Andrej Dujella, Matija Kazalicki, Vinko Petričević

Abstract

A set of m distinct nonzero rationals $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$, is called a rational Diophantine m-tuple. It is proved recently that there are infinitely many rational Diophantine sextuples. In this paper, we construct infinite families of rational Diophantine sextuples with special structure, namely the sextuples containing quadruples and quintuples of certain type.

Keywords: rational Diophantine sextuples, regular Diophantine quadruples, regular Diophantine quintuples, elliptic curves
2010 Math. Subj. Class.: Primary 11D09; Secondary 11G05

1 Introduction

A Diophantine m-tuple is a set of m distinct positive integers with the property that the product of any two of its distinct elements plus 1 is a square. Fermat found the first Diophantine quadruple in integers $\{1, 3, 8, 120\}$. If a set of m nonzero rationals has the same property, then it is called a rational Diophantine m-tuple. The first example of a rational Diophantine quadruple was the set

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$$

found by Diophantus. Euler proved that the exist infinitely many rational Diophantine quintuples (see [17]), in particular he was able to extend the integer Diophantine quadruple found by Fermat, to the rational quintuple

$$\left\{1, 3, 8, 120, \frac{777480}{8288641}\right\}.$$

Stoll [20] recently showed that this extension is unique. Therefore, the Fermat set $\{1, 3, 8, 120\}$ cannot be extended to a rational Diophantine sextuple.

In 1969, using linear forms in logarithms of algebraic numbers and a reduction method based on continued fractions, Baker and Davenport [1] proved that if d is a positive integer such that $\{1, 3, 8, d\}$ forms a Diophantine quadruple, then d has to be 120. This result motivated the conjecture that there does not exist a Diophantine quintuple in integers. The conjecture has been proved recently by He, Togbé and Ziegler [16] (see also [2, 7]).

In the other hand, it is not known how large can be a rational Diophantine tuple. In 1999, Gibbs found the first example of rational Diophantine sextuple [13]

$$\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}.$$

In 2017 Dujella, Kazalicki, Mikić and Szikszai [11] proved that there are infinitely many rational Diophantine sextuples, while Dujella and Kazalicki [10] (inspired by the work of Piezas [19]) described another construction of parametric families of rational Diophantine sextuples. Recently, Dujella, Kazalicki and Petričević [12] proved that there are infinitely many rational Diophantine sextuples such that denominators of all the elements (in the lowest terms) in the sextuples are perfect squares. No example of a rational Diophantine septuple is known. The Lang conjecture on varieties of general type implies that the number of elements of a rational Diophantine tuple is bounded by an absolute constant (see Introduction of [11]). For more information on Diophantine m-tuples see the survey article [9].

Although the constructions of infinite families of rational Diophantine sextuples in [10] and [11] are essentially different, they have one common feature. Namely, in both constructions (and also in [12], which is a special case of [10]) the sextuples contain two regular rational Diophantine quintuples. The quintuple $\{a, b, c, d, e\}$ is called regular if

$$(abcde + 2abc + a + b + c - d - e)^2 = 4(ab + 1)(ac + 1)(bc + 1)(de + 1)$$

(see [4, 6, 8, 14]). Similarly, the quadruple $\{a, b, c, d\}$ is called regular if

$$(a + b - c - d)^2 = 4(ab + 1)(cd + 1)$$

(see [6] for characterization of regular Diophantine quadruples and quintuples in terms of corresponding elliptic curves).

In [15], Gibbs collected over 1000 examples of rational Diophantine sextuples with relatively small numerators and denominators. These examples are also sorted in [15] according to their structure, which includes information of regular quadruples and quintuples which they contain. We have extended the search for sextuples with small numerators and denominators and included also examples with mixed signs (in [15] only sextuples with positive elements were considered). We have observed a significant number of sextuples which contain exactly one regular Diophantine quintuple and two regular Diophantine quadruples. Thus, in this paper we study rational Diophantine sextuples having this structure. Our main result is the following theorem.

Theorem 1. *There are infinitely many rational Diophantine sextuples which contain one regular Diophantine quintuple and two regular Diophantine quadruples.*

2 Parametrizations of Diophantine triples

Let $\{a_1, a_2, a_3\}$ be a rational Diophantine triple and let $a_2 = (r^2 - 1)/a_1$ and $a_3 = (s^2 - 1)/a_1$ for rationals r and s. By putting $a_2 a_3 + 1 = (a_2 s^2 - a_2 + a_1)/a_1 = (1 + (s - 1)t)^2$, we get

$$a_3 = \frac{-4t(t-1)(a_1 t - a_2)}{(-a_2 + a_1 t^2)^2}.$$

This parametrization of Diophantine triples was used in [8] in construction of certain rational Diophantine sextuples. Here we will use an equivalent, but simpler and more aesthetic parametrization due to Lasić [18], which is symmetric in the three involved parameters:

$$\begin{aligned} a_1 &= \frac{2t_1(1 + t_1 t_2(1 + t_2 t_3))}{(-1 + t_1 t_2 t_3)(1 + t_1 t_2 t_3)}, \\ a_2 &= \frac{2t_2(1 + t_2 t_3(1 + t_3 t_1))}{(-1 + t_1 t_2 t_3)(1 + t_1 t_2 t_3)}, \\ a_3 &= \frac{2t_3(1 + t_3 t_1(1 + t_1 t_2))}{(-1 + t_1 t_2 t_3)(1 + t_1 t_2 t_3)}. \end{aligned}$$

The connection between two parametizations is given by

$$t_1 = \frac{a_1}{r-1}, \quad t_2 = \frac{-(1 - r^2 + a_1^2 t^2)}{2(t-1)a_1}, \quad t_3 = \frac{2a_1 t(t-1)}{1 - r^2 + a_1^2 t^2},$$

$$a_1 = \frac{2t_1(1+t_1t_2(1+t_2t_3))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)},$$
$$r = \frac{1+2t_1t_2+2t_1t_2^2t_3+t_2^2t_3^2t_1^2}{(-1+t_1t_2t_3)(1+t_1t_2t_3)}, \quad t = -t_2t_3.$$

3 New construction of families of Diophantine sextuples

Let $\{a_1, a_2, a_3, a_4\}$ and $\{a_1, a_2, a_3, a_5\}$ be regular Diophantine quadruples, i.e. a_4 and a_5 are solutions of the quadratic equation

$$(a_1 + a_2 - a_3 - x)^2 - 4(a_1a_2 + 1)(a_3x + 1) = 0.$$

We obtain that

$$a_4 = \frac{-2(1-t_3+t_2t_3)(t_3t_1+1-t_1)(-t_2+1+t_1t_2)(-1+t_1t_2t_3)}{(1+t_1t_2t_3)^3},$$
$$a_5 = \frac{2(t_3+t_2t_3+1)(t_3t_1+t_1+1)(1+t_2+t_1t_2)(1+t_1t_2t_3)}{(-1+t_1t_2t_3)^3}.$$

In order that $\{a_1, a_2, a_3, a_4, a_5\}$ be a rational Diophantine quintuple, it remains to satisfy the condition that $a_4a_5 + 1$ is a perfect square. We obtain the condition that

$$\begin{aligned}
p(t_1,t_2,t_3) = {} & (-8t_2^3t_3^3 - 8t_3^2t_2^2 - 3t_3^4t_2^4 + 4t_2^2 + 4t_2^2t_3^4 + 4t_2^4t_3^2 + 8t_2^3t_3)t_1^4 \\
& + (8t_2^2t_3 - 16t_2t_3^2 - 8t_2^3t_3^2 + 8t_2 - 8t_2^3t_3^4 - 8t_2^4t_3^3 - 8t_2^2t_3^3 + 8t_3^4t_2)t_1^3 \\
& + (-8t_3^2 - 8t_2^2 - 8t_2t_3 - 8t_2^3t_3^3 - 8t_2^2t_3^4 + 8t_3^3t_2 + 4t_3^4 \\
& + 4 - 18t_3^2t_2^2 + 4t_3^4t_2^4 - 8t_2^4t_3^2 - 16t_2^3t_3)t_1^2 \\
& + (8t_2^4t_3^3 - 8t_2^2t_3 - 16t_2^2t_3^3 - 8t_2t_3^2 - 8t_2 + 8t_3^3 + 8t_2^3t_3^2 - 8t_3)t_1 \\
& - 3 - 8t_2t_3 + 4t_2^4t_3^2 - 8t_3^2t_2^2 + 4t_3^2 + 4t_2^2 + 8t_2^3t_3
\end{aligned} \qquad (1)$$

is a perfect square. We compute the discriminant of the quartic polynomial p with the respect to t_1 and factorize it. One of the factors is

$$p_1(t_2, t_3) = 3 + 10t_2t_3 - 3t_3^2 + 3t_3^2t_2^2$$

(other factors either have much larger degree or correspond to quintuples with an element equal to 0). The condition $p_1(t_2, t_3) = 0$ (which ensures that the polynomial p with the respect of t_1 has a double root) leads to $9t_3^2 + 16$ be a perfect square, say $9t_3^2 + 16 = (3t_3 + u)^2$. We get

$$t_3 = \frac{16 - u^2}{6u},$$

$$t_2 = \frac{u^2 + 10u + 16}{(u-4)(u+4)}.$$

Inserting this in (1), we obtain that a_4a_5+1 is a perfect square. Thus, we obtained a two-parametric family (in parameters t_1 and u) of rational Diophantine quintuples which contain two regular quadruples (let us mention that a one-parametric family of rational Diophantine quintuples with this property was constructed in [5]).

Now we extend the nonregular quadruple $\{a_1, a_3, a_4, a_5\}$ (analogously we could choose any of other two nonregular quadruples contained in the quintuple $\{a_1, a_2, a_3, a_4, a_5\}$) to regular quintuples $\{a_1, a_3, a_4, a_5, a_6\}$ and $\{a_1, a_3, a_4, a_5, a_7\}$, i.e. a_6 and a_7 are solutions of the quadratic equation

$$\begin{aligned}(a_1a_3a_4a_5x + 2a_1a_3a_4 + a_1 + a_3 + a_4 - a_5 - x)^2 \\ - 4(a_1a_3+1)(a_1a_4+1)(a_3a_4+1)(a_5x+1) = 0.\end{aligned}$$

We will not use a_7 is our construction, so we give here only the value of a_6:

$$\begin{aligned}a_6 = 6(u+4)(u+8)(u+2)(u-4)(2t_1u^2 + 3u^2 + 20t_1u + 12u + 32t_1) \\ \times (t_1u^2 + 10t_1u + 16t_1 - 6u)(t_1u^2 + 10t_1u + 16t_1 + 6u) \\ \times (t_1u^2 + 10t_1u + 16t_1 - 24 - 6u) \\ \times (4096t_1^2 + 15360t_1^2u + 15168t_1^2u^2 + 5920t_1^2u^3 + 948t_1^2u^4 + 60t_1^2u^5 + t_1^2u^6 \\ -12288t_1u - 7680t_1u^2 + 480t_1u^4 + 48t_1u^5 - 5184u^2 - 2592u^3 - 324u^4)^{-2}.\end{aligned}$$

The only missing condition in order that $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ be a rational Diophantine sextuple is that a_2a_6+1 is a perfect square. This condition leads to the quartic in t_1 over $\mathbb{Q}(u)$:

$$\begin{aligned}&(u^{12} + 120u^{11} + 5496u^{10} + 125600u^9 + 1639440u^8 \\ &\quad + 13075200u^7 + 65656320u^6 + 209203200u^5 + 419696640u^4 \\ &\quad + 514457600u^3 + 360185856u^2 + 125829120u + 16777216)t_1^4 \\ &+ (24u^{12} + 1296u^{11} + 32256u^{10} + 446208u^9 + 3461760u^8 \\ &\quad + 13047552u^7 - 208760832u^5 - 886210560u^4 - 1827667968u^3 \\ &\quad - 2113929216u^2 - 1358954496u - 402653184)t_1^3 \\ &+ (36u^{12} + 1296u^{11} + 18072u^{10} + 48096u^9 - 1681632u^8 \\ &\quad - 22516992u^7 - 127051776u^6 - 360271872u^5 - 430497792u^4 \\ &\quad + 197001216u^3 + 1184366592u^2 + 1358954496u + 603979776)t_1^2 \\ &+ (- 432u^{11} - 15552u^{10} - 259200u^9 - 2267136u^8 - 9116928u^7 + 145870848u^5 \\ &\quad + 580386816u^4 + 1061683200u^3 + 1019215872u^2 + 452984832u)t_1\end{aligned}$$

$$+1296u^{10}+41472u^9+31643136u^6+670032u^8+6054912u^7$$
$$+96878592u^5+171528192u^4+169869312u^3+84934656u^2=z^2.$$

Since this quartic has a $\mathbb{Q}(u)$-rational point at infinity, it can be transformed by birational transformations into an elliptic curve over $\mathbb{Q}(u)$, see e.g. [3, Section 1.2] (the singular point at infinity on the quartic corresponds to the point at infinity and an additional point P_1 on the elliptic curve). The quartic has another $\mathbb{Q}(u)$-rational point corresponding to $t_1=\frac{-3(u+4)u}{2(u^2+10u+16)}$. It gives $a_6=0$, so it does not yield a rational Diophantine sextuples. However, if we denote the corresponding point on the elliptic curve by P_2, then the point $2P_2$ on the elliptic curve corresponds to the point with

$$t_1=\frac{3(3u^4+40u^3+368u^2+1280u+1024)}{4(u^2+10u+16)(u+20)u}$$

on the quartic, and by inserting this value, we obtain the parametric family of rational Diophantine sextuples

$$\Big\{\frac{-12u(u+4)(3u^4+8u^3+224u^2+576u+512)(3u^3+28u^2+256u+256)}{(u+8)(u+2)(u-4)(3u^3+8u^2+144u+128)(3u^4+48u^3+528u^2+1280u+1024)},$$
$$\frac{8u(u+20)(3u^5+8u^4+64u^3-640u^2-2304u-2048)(u+8)(u+2)}{3(u+4)(u-4)(3u^3+8u^2+144u+128)(3u^4+48u^3+528u^2+1280u+1024)},$$
$$\frac{2(u+4)(u-4)}{3(u+8)(u+2)(3u^3+8u^2+144u+128)}\times\frac{39u^7+776u^6+8096u^5+48640u^4+226048u^3+587776u^2+770048u+393216}{3u^4+48u^3+528u^2+1280u+1024},$$
$$\frac{-8(u^2+4u+32)(3u^3+14u^2-40u-64)(9u^3+8u^2+112u+384)}{3(u+8)(u+4)(u+2)(u-4)}\times\frac{3u^4+48u^3+528u^2+1280u+1024}{(3u^3+8u^2+144u+128)^3},$$
$$\frac{4u(u+2)(17u^2+48u+48)(3u^5+8u^4-176u^3-2944u^2-9216u-8192)}{3(u+4)(u-4)}\times\frac{(3u^3+8u^2+144u+128)(u+8)^2}{(3u^4+48u^3+528u^2+1280u+1024)^3},$$
$$\frac{12(u+2)(u-4)(5u+8)(u+4)(3u^2+8u+64)}{u+8}\times\frac{(3u^3+8u^2+144u+128)(3u^4+48u^3+528u^2+1280u+1024)}{(16384+69632u+64768u^2+22272u^3+3680u^4+576u^5+9u^6)^2}\Big\}$$

which satisfies the properties from Theorem 1.

E.g. for $u=-1$ we get the rational Diophantine sextuple

$$\left\{\frac{27900}{17479},\frac{471352}{112365},\frac{261770}{17479},\frac{185535272}{419265},\frac{63737828}{526368735},\frac{79554420}{408480247}\right\}.$$

By taking other linear combinations of the points P_1 and P_2 we can also obtain (more complicated) families of rational Diophantine sextuples.

Acknowledgements. The authors would like to thank the referee for careful reading of this paper and his useful comments. The authors were supported by the Croatian Science Foundation under the project no. IP-2018-01-1313. The authors acknowledge support from the QuantiXLie Center of Excellence, a project co-financed by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004). The authors acknowledge the usage of the supercomputing resources of Division of Theoretical Physics at Ruđer Bošković Institute, as well as the computing resources at Department of Mathematics, University of Zagreb which were provided by Croatian Science Foundation grant HRZZ-9345

References

[1] A. Baker and H. Davenport, *The equations* $3x^2-2=y^2$ *and* $8x^2-7=z^2$, Quart. J. Math. Oxford Ser. (2) **20** (1969), 129–137.

[2] M. Bliznac Trebješanin and A. Filipin, *Nonexistence of* $D(4)$*-quintuples*, J. Number Theory **194** (2019), 170–217.

[3] I. Connell, Elliptic Curve Handbook, McGill University, Montreal, 1999.

[4] A. Dujella, *On Diophantine quintuples*, Acta Arith. **81** (1997), 69–79.

[5] A. Dujella, *Diophantine triples and construction of high-rank elliptic curves over* $\mathbb{Q}$ *with three non-trivial* 2*-torsion points*, Rocky Mountain J. Math. **30** (2000), 157–164.

[6] A. Dujella, *Irregular Diophantine m-tuples and elliptic curves of high rank*, Proc. Japan Acad. Ser. A Math. Sci. **76** (2000), 66–67.

[7] A. Dujella, *There are only finitely many Diophantine quintuples*, J. Reine Angew. Math. **566** (2004), 183–214.

[8] A. Dujella, *Rational Diophantine sextuples with mixed signs*, Proc. Japan Acad. Ser. A Math. Sci. **85** (2009), 27–30.

[9] A. Dujella, *What is ... a Diophantine m-tuple?*, Notices Amer. Math. Soc. **63** (2016), 772–774.

[10] A. Dujella and M. Kazalicki, *More on Diophantine sextuples*, in: Number Theory - Diophantine problems, uniform distribution and applications, Festschrift in honour of Robert F. Tichy's 60th birthday (C. Elsholtz, P. Grabner, Eds.), Springer-Verlag, Berlin, 2017, pp. 227–235.

[11] A. Dujella, M. Kazalicki, M. Mikić and M. Szikszai, *There are infinitely many rational Diophantine sextuples*, Int. Math. Res. Not. IMRN **2017 (2)** (2017), 490–508.

[12] A. Dujella, M. Kazalicki and V. Petričević, *There are infinitely many rational Diophantine sextuples with square denominators*, J. Number Theory **205** (2019), 340–346.

[13] P. Gibbs, *Some rational Diophantine sextuples*, Glas. Mat. Ser. III **41** (2006), 195–203.

[14] P. Gibbs, *Regular rational Diophantine sextuples*, preprint, 2016.

[15] P. Gibbs, *A survey of rational diophantine sextuples of low height*, preprint, 2016.

[16] B. He, A. Togbé and V. Ziegler, *There is no Diophantine quintuple*, Trans. Amer. Math. Soc. **371** (2019), 6665–6709.

[17] T. L. Heath, Diophantus of Alexandria. A Study in the History of Greek Algebra. Powell's Bookstore, Chicago; Martino Publishing, Mansfield Center, 2003.

[18] L. Lasić, personal communication, 2017.

[19] T. Piezas, *Extending rational Diophantine triples to sextuples*, `http://mathoverflow.net/questions/233538/extending-rational-diophantine-triples-to-sextuples`

[20] M. Stoll, *Diagonal genus 5 curves, elliptic curves over $\mathbb{Q}(t)$, and rational diophantine quintuples*, Acta Arith. **190** (2019), 239–261.

Andrej Dujella
Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia
E-mail address: `duje@math.hr`

Matija Kazalicki
Department of Mathematics, Faculty of Science, University of Zagreb,

Bijenička cesta 30, 10000 Zagreb, Croatia
E-mail address: matija.kazalicki@math.hr

Vinko Petričević
Department of Mathematics, Faculty of Science, University of Zagreb,
Bijenička cesta 30, 10000 Zagreb, Croatia
E-mail address: vpetrice@math.hr