ACTA MATHEMATICA SPALATENSIA
Vol.1 (2021) 1–18
https://doi.org/10.32817/ams.1.1.1

Received: 23 May 2017
Accepted: 06 Dec 2017

A note on curves

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Abstract

There are several rather different approaches to the notion of a curve. We have found the way which assures that, beside an arc, each circle carries the unique curve. As a consequence, each 1-parametrizable set yields at most finitely many curves. Further, all essential properties and well known invariants of a curve are preserved.

Keywords: 1-parametrizable set, curve, oriented curve
2010 Math. Subj. Class.: Primary 53A04; Secondary 26B12

1 Introduction

In the mathematical literature one can find several rather different approaches to the notion of a curve. Even more, in spite of the very similar intention and purposes, the name *curve* covers several different notions. For instance,

- a *curve* is a mapping (continuous function) $\gamma\colon [\alpha,\beta] \to \mathbb{R}^k$ ([4, p. 136, 6.26 Definition]);
- a *curve* in a space Y is the image of a mapping $r\colon [\alpha,\beta] \to Y$ ([2, p. 105]);
- an *algebraic* or *transcendent curve* is a graph of an appropriate implicit function ([1, pp. 404, 476]);
- a *piecewise smooth curve* is a "union" of finitely many smooth simple curves, where a *smooth simple curve* is ordered pair (Γ, r),

$r\colon [\alpha,\beta] \to \Gamma \subseteq E = \mathbb{R}^k$ is a continuously differentiable bijection with every $r'(t) \neq 0$ ([3, pp. 171–173]);

- a *curve* in $\mathbb{R}^n$ is an ordered pair $(\Gamma, [r])$, where $r\colon [\alpha,\beta] \to \Gamma \subseteq \mathbb{R}^n$ is a continuous surjection having the singular set finite and $[r]$ is a certain equivalence class of r ([5, p. 253]; see also Section 2 below).

In the most general (mapping) case, the notions of a curve and a path are identified. Further, since in that, as well as in the image case, there are "spacefilling" (Peano) curves, the notion contradicts to the "natural" understanding of a curve. In the case of a graph, the notion of a curve is reduced to a specific set (which, in general, is not a continuum), that is unsatisfactory for many purposes. The approach by means of an ordered pair consisting of a set and a (class of) mapping(s) seems to be adequate in mathematical analysis. However, it still admits to many curves on some types of given sets, when one expects just a few or a unique one. In this paper we have succeeded to find an equivalence relation that generally solves the problem in affirmative (Section 3).

2 Preliminaries

The needed simple topological facts are well known and can be found in [2]. Let us firstly recall the notion of a parametrizable set ([5, 4. Definition 29.1; p. 252]). A subspace $\Gamma \subseteq \mathbb{R}^n$ is said to be a 1-*parametrizable set* (or one says that Γ *admits a* 1-*parametrization*), if there exists a surjective mapping (*continuous* function)

$$r\colon [\alpha,\beta] \to \Gamma,\ \alpha < \beta,$$

having the singular set

$$S(r) \equiv \left\{\tau \in [\alpha,\beta] \mid r^{-1}[\{r(\tau)\}] \neq \{\tau\}\right\}$$

finite. In that case, r is said to be a (1-)*parametrization of* Γ, and the ordered pair (Γ, r) is called a (1-)*parametrized set.*

Example 1. *One can easily verify that each circle $S^1 \subseteq \mathbb{R}^n$, $n \geq 2$, is a parametrizable set (in many different ways), while the* Hawaiian earring, *i.e., the subspace*

$$H = \cup_{k\in\mathbb{N}} S^1_k \subseteq \mathbb{R}^2, \quad S^1_k \equiv \left\{(\xi,\eta) \in \mathbb{R}^2 \mid \xi^2 + (\eta + \frac{1}{k})^2 = \frac{1}{k^2}\right\},$$

does not admit any parametrization. Further, a circle with m segments attached at the same point is a 1-*parametrizable set, if and only if $m \leq 2$, while a circle with m segments attached, each at a different point, is a* 1-*parametrizable set, if and only if $m \leq 1$.*

Remark 2. *Observe that, for every parametrization,* $r\colon [\alpha,\beta] \to \Gamma$*, the restrictions*

$$r|\,\langle \tau_{k-1}, \tau_k\rangle : \langle \tau_{k-1}, \tau_k\rangle \to r[\langle \tau_{k-1}, \tau_k\rangle], \quad k = 1, \dots, l+1,$$

are homeomorphisms, where

$$\{\tau_1, \dots, \tau_l\} = S(r),\ \tau_0 \equiv \alpha \leq \tau_1 < \cdots < \tau_l \leq \beta \equiv \tau_{l+1}.$$

Furthermore, by the general topology facts, every parametrizable set $\Gamma \subseteq \mathbb{R}^n$ *is a metric continuum (i.e., a compact and connected metric space) having dimension* $\dim \Gamma = 1$*.*

By following [5, 4. Definition 29.2 (p. 252)], let $\Gamma \subseteq \mathbb{R}^n$ be a 1-parametrizable set, and let

$$r^i\colon [\alpha_i, \beta_i] \to \Gamma,\ i = 1, 2,$$

be a pair of its parametrizations. Then r^2 is said to be *compatible* with (or *equivalent* to) r^1 if there exists a strictly monotone (either increasing or decreasing) mapping

$$\omega\colon [\alpha_1, \beta_1] \to [\alpha_2, \beta_2]$$

such that $r^2\omega = r^1$. Given a 1-parametrizable set $\Gamma \subseteq \mathbb{R}^n$, one straightforwardly verifies that being compatible is an equivalence relation on the set of all paramatrizations of Γ. Denoting by $\langle r\rangle$ the compatibility (equivalence) class of a parametrization r of Γ, the "standard" definition of a (geometric) curve in the euclidean space $\mathbb{R}^n$ reads as follows (compare [5, 4. Definition 29.3; p. 253]): A *(standard, geometric) curve* C in $\mathbb{R}^n$ is an ordered pair $(\Gamma, \langle r\rangle)$, where $\Gamma \subseteq \mathbb{R}^n$ is a 1-parametrizable set, and $\langle r\rangle$ is the compatibility class of a parametrization $r\colon [\alpha,\beta] \to \Gamma$. It is well known that every homeomorphic image $\Gamma \subseteq \mathbb{R}^n$ of a segment $[\alpha,\beta] \subseteq \mathbb{R}$ is a 1-parametrizable set having all the parametrizations mutually compatible. The corresponding unique curve is called a *simple curve* or an *arc*. However, since the continuity and strict monotonicity conditions on ω are very restrictive, in the case of a 1-parametrizable set making a loop, the situation changes drastically.

Example 3. *Let*

$$\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C} = \mathbb{R}^2$$

be the central unit circle (the standard 1*-sphere). Let the function*

$$r\colon [0, 2\pi] \to \mathbb{S}^1$$

be defined by

$$r(\tau) = (\cos\tau, \sin\tau) = e^{i\tau}.$$

Then r is a continuous surjection having $S(r) = \{0, 2\pi\}$, and thus, $\mathbb{S}^1$ is a parametrizable set and r is a parametrization of $\mathbb{S}^1$. Further, the function

$$p \colon [0, 2\pi] \to \mathbb{S}^1,$$

defined by

$$p(\tau) = (\cos(\tau + \pi), \sin(\tau + \pi)) = e^{i(\tau+\pi)},$$

is a parametrization of $\mathbb{S}^1$ as well (having the same singular set $\{0, 2\pi\}$). An easy analysis shows that, for every strictly monotone mapping

$$\omega \colon [0, 2\pi] \to [0, 2\pi],$$

$p\omega \neq r$ holds. Thus, we have obtained two different (standard) curves, $(\mathbb{S}^1, \langle r \rangle)$ and $(\mathbb{S}^1\ \langle p \rangle)$, on the the same circle $\mathbb{S}^1$. Moreover, one readily sees that $\mathbb{S}^1$ carries ***uncountable many*** *mutually different (standard) curves. (Notice that the function*

$$q \colon [-\pi, \pi] \to \mathbb{S}^1,$$

defined by

$$q(\tau) = (\cos(\tau + \pi), \sin(\tau + \pi)) = e^{i(\tau+\pi)},$$

is also a parametrization of $\mathbb{S}^1$, which is compatible with r, because

$$\rho \colon [0, 2\pi] \to [-\pi, \pi], \ \rho(\tau) = \tau - \pi,$$

is a strictly increasing mapping and $q\rho = r$. Thus, the choice of a segment $[\alpha, \beta]$ is relevant too!)

The natural questions arises: Is it possible to modify the definition of a (standard, geometric) curve such that, beside an arc, **each circle yields the unique curve**, and that all essential properties and invariants of a curve are preserved? In this paper we answer the question in affirmative.

3 The new classification of parametrizations

In order to answer the stated question in affirmative, we are looking for a suitable classification of parametrizations that has to be strictly coarser than the compatibility. First of all, recall that a function

$$w \colon [\alpha, \beta] \to Y \subseteq \mathbb{R}$$

is said to be piecewise (strictly) monotone if there are (finitely many) points

$$\xi_1, \dots, \xi_n \in \langle \alpha, \beta \rangle \,, \; \xi_0 \equiv \alpha < \xi_1 < \cdots < \xi_n < \beta \equiv \xi_{n+1},$$

i.e., if there is a finite subdivision of $[\alpha, \beta]$, such that all the restrictions (onto the open intervals yielded by the subdivision)

$$w| \langle \xi_{j-1}, \xi_j \rangle : \langle \xi_{j-1}, \xi_j \rangle \to Y, \; j = 1, \dots, n+1,$$

are (strictly) monotone. Especially, if all the restrictions are strictly increasing (strictly decreasing), then w is said to be piecewise strictly increasing (piecewise strictly decreasing). If, in addition, for such a function

$$w \colon [\alpha, \beta] \to [\gamma, \delta] \subseteq \mathbb{R}$$

(either piecewise strictly increasing or piecewise strictly decreasing), each restriction $w| \langle \xi_{j-1}, \xi_j \rangle$, $j = 1, \dots, n+1$, is continuous, then w is said to be a *parametric substitute.*

3.1 The curve

Definition 4. *Let $\Gamma \subseteq \mathbb{R}^n$ be a 1-parametrizable set, and let*

$$r^i \colon [\alpha_i, \beta_i] \to \Gamma, \; i = 1, 2,$$

be a pair of its parametrizations. Then r^2 is said to be ***comparable*** *to r^1, if there exists a parametric substitute*

$$w \colon [\alpha_1, \beta_1] \to [\alpha_2, \beta_2]$$

such that $r^2 w = r^1$.

Remark 5. *It suffices to require, in Definition 4, that w is a function which is either piecewise strictly increasing or piecewise strictly decreasing. Namely, in that case, the continuity (and bijectivity) of all $w| \langle \xi_{j-1}, \xi_j \rangle$, $j = 1, \dots, n+1$, follows straightforwardly by $r^2 \omega = r^1$ and Remark 2. Consequently, the following fact holds:*
If r^1, r^2 are parametrizations of Γ and w_1, w_2 are piecewise strictly monotone functions such that $r^2 w_1 = r^1 = r^2 w_2$, then

$$w_1 | ([\alpha_1, \beta_1] \setminus (r^1)^{-1}[r^2[S(r^2)]]) = w_2 | ([\alpha_1, \beta_1] \setminus (r^1)^{-1}[r^2[S(r^2)]]).$$

In other words, a piecewise strictly monotone function

$$w \colon [\alpha_1, \beta_1] \to [\alpha_2, \beta_2]$$

(not necessarily either piecewise strictly increasing or piecewise strictly decreasing) satisfying $r^2 w = r^1$ is unique up to the finite set

$$(r^1)^{-1}[r^2[S(r^2)]] \subseteq [\alpha_1, \beta_1]$$

and, moreover, its restriction to each (open) interval $\langle \xi_{j-1}, \xi_j \rangle$, determined by that set and $S(r^1)$, is a homeomorphism.

Lemma 6. *Given a 1-parametrizable set $\Gamma \subseteq \mathbb{R}^n$, the comparability is an equivalence relation on the set of all parametrizations r of Γ. The comparability (equivalence) class of an r is denoted by $[r]$.*

Proof. Let $r\colon [\alpha, \beta] \to \Gamma$ be a parametrization of Γ. Since the identity $1_{[\alpha,\beta]}$ is strictly increasing and $r 1_{[\alpha,\beta]} = r$, the comparability is a reflexive relation. Let $p\colon [\gamma, \delta] \to \Gamma$ be a parametrization of Γ that is comparable to r, i.e., $pw = r$, for some $w\colon [\alpha, \beta] \to [\gamma, \delta]$ according to Definition 4. Put $w^-\colon [\gamma, \delta] \to [\alpha, \beta]$ to be the inverse of w on the image $\langle \eta_{j-1}, \eta_j \rangle$ of each maximal interval $\langle \xi_{j-1}, \xi_j \rangle$ of the strict monotonicity of w, while on the remaining (finitely many) points choose the values $w^-(\eta_j)$ according to commutativity $r(w^-(\eta_j)) = p(\eta_j)$. Then w^- inherits the monotonicity properties of w and $rw^- = p$ holds. Thus, r is comparable to p, and the symmetry is proven. Finally, let a parametrization $q\colon [\epsilon, \kappa] \to \Gamma$ be comparable to p and let p be comparable to r, i.e., $pw_1 = r$ and $qw_2 = p$ according to Definition 4. Put $w\colon [\alpha, \beta] \to [\epsilon, \kappa]$ to be the composite $w_2 w_1$ on the appropriate maximal intervals of its strict monotonicity, while on the remaining (finitely many) points choose the values $w(\xi_j)$ according to commutativity $q(w(\xi_j)) = r(\xi_j)$. Then $qw = r$ holds true. Moreover, if w_1 and w_2 are of the same "monotonicity kind", then w is piecewise strictly increasing, while if w_1 and w_2 are of the different (opposite) "monotonicity kind", then w is piecewise strictly decreasing. Consequently, r is comparable to q which proves the needed transitivity, and completes the proof of the lemma. □

Definition 7. *A* ***curve*** *in $\mathbb{R}^n$ is every ordered pair $(\Gamma, [r])$ consisting of a 1-parametrizable set $\Gamma \subseteq \mathbb{R}^n$ and the comparability (equivalence) class $[r]$ of a parametrization $r\colon [\alpha, \beta] \to \Gamma$ of Γ.*

Clearly, a parametrizable set can, generally, yield more than one curve (see Example 9 and Corollary 14 below). Nevertheless, we primarily want to characterize those which admit the unique curves, i.e., those having all the parametrizations comparable.

Lemma 8. *Given a 1-parametrizable set $\Gamma \subseteq \mathbb{R}^n$, let*

$$r^i\colon [\alpha, \beta] \to \Gamma, \ i = 1, 2,$$

be a pair of its parametrizations. If $S(r^1), S(r^2) \subseteq \{\alpha, \beta\}$, then r^1 and r^2 are comparable.

Proof. Observe, firstly, that $S(r^1), S(r^2) \subseteq \{\alpha, \beta\}$ implies $S(r^1) = S(r^2)$. In that case, indeed, either $S(r^i) = \varnothing$ or $S(r^i) = \{\alpha, \beta\}$, $i = 1, 2$. Namely, a singular set cannot be a singleton. Now, assume to the contrary, i.e., that either $S(r^1) = \varnothing$ and $S(r^2) = \{\alpha, \beta\}$, or $S(r^1) = \{\alpha, \beta\}$ and $S(r^2) = \varnothing$. In the first case, by Definition 4, r^1 is a homeomorphism (a continuous bijection on the compactum $[\alpha, \beta]$), and thus, $\Gamma \approx [\alpha, \beta]$, while r^2 is a "homeomorphism up to $\{\alpha, \beta\}$" with $r^2(\alpha) = r^2(\beta)$, and thus, $\Gamma \approx \mathbb{S}^1$ (the standard circle). Consequently, $\mathbb{S}^1 \approx [\alpha, \beta]$ - a contradiction. The second case leads to the same contradiction.
Suppose that $S(r^1) = S(r^2) = \emptyset$. Then r^1 and r^2 are homeomorphisms, and thus the composite

$$w \equiv (r^2)^{-1} r^1 \colon [\alpha, \beta] \to [\alpha, \beta]$$

is a homeomorphism. Since every bijection on a segment is strictly monotone, it follows that w is a parametric substitute. Finally, by Lemma 6 and

$$r^2 w = r^2 (r^2)^{-1} r^1 = r^1,$$

it follows that the parametrizations r^1 and r^2 are comparable.
Suppose that $S(r^1) = S(r^2) = \{\alpha, \beta\}$. Then, by the definition and Remark 2, $r^i(\alpha) = r^i(\beta) \equiv x_i \in \Gamma$, $i = 1, 2$, and the restrictions

$$r^i | \langle \alpha, \beta \rangle : \langle \alpha, \beta \rangle \to \Gamma \setminus \{x_i\}, \ i = 1, 2,$$

are homeomorphisms, i.e., Γ is homeomorphic to the standard circle. This means that r^i, $i = 1, 2$, "winds up" the segment $[\alpha, \beta]$ onto Γ in a unique of two possible ways: either "clockwise (-)" or "counterclockwise (+)". Namely, if it is not so, then the restriction $r^i | \langle \alpha, \beta \rangle$ is not bijective or it is not continuous - a contradiction. Now the following two cases may occur: either $x_1 = x_2 \equiv x_0$ or $x_1 \neq x_2$. If $x_1 = x_2$, then the composite

$$\phi \equiv (r^2)^{-1} | (\Gamma \setminus \{x_0\})) \circ (r^1 | \langle \alpha, \beta \rangle) \colon \langle \alpha, \beta \rangle \to \langle \alpha, \beta \rangle$$

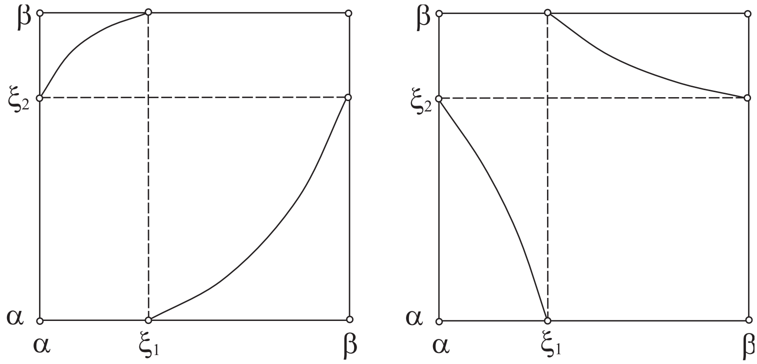
is well defined. Moreover, ϕ is a continuous bijection, and thus, it is strictly monotone. Put

$$w \colon [\alpha, \beta] \to [\alpha, \beta]$$

to be the (unique) monotone extension of ϕ by values $w(\alpha), w(\beta) \in \{\alpha, \beta\}$. More precisely, among four possibilities, only two of them yield a bijective function w, and only one of these is monotone (and continuous). Therefore, w is a parametric substitute and, clearly, $r^2 w = r^1$, which shows that the parametrizations r^1 and r^2 are comparable. If $x_1 \neq x_2$, then there exists a unique pair $\xi_1, \xi_2 \in \langle\alpha, \beta\rangle$ such that $r^1(\xi_1) = x_2$ and $r^2(\xi_2) = x_1$. Consider the composite

$$\psi \equiv (r^2)^{-1}|(\Gamma\backslash\{x_1, x_2\}))\circ(r^1|\,\langle\alpha, \beta\rangle\backslash\{\xi_1\})\colon\ \langle\alpha, \beta\rangle\backslash\{\xi_1\} \to \langle\alpha, \beta\rangle\backslash\{\xi_2\}.$$

It means that $\psi(\tau) = \tau'$ if and only if, $r^1(\tau) = r^2(\tau')$ and $\tau \notin \{\alpha, \xi_1, \beta\}$, $\tau' \notin \{\alpha, \xi_2, \beta\}$. According to the above mentioned "winding up", one readily sees that graph of ψ can be one of the following two (typical) only:



Namely, among four possibilities in total, i.e., all the 2-combinations of the $\{(r^1)^{\mp}, (r^2)^{\mp}\}$ with respect to the "$\mp$ winding up", the pairs $(-,-)$ and $(+,+)$ yield a piecewise strictly increasing function, while the pairs $(-,+)$ and $(+,-)$ yield a piecewise strictly decreasing function. Consequently, by choosing an extension w of ψ onto $[\alpha, \beta]$ such that

$$w\colon [\alpha, \beta] \to [\alpha, \beta],\ w(\alpha) = w(\beta) = \xi_2,\ w(\xi_1) \in \{\alpha, \beta\}$$

(one of two possible), we obtain a parametric substitute such that $r^2 w = r^1$. This shows that the parametrizations r^1 and r^2 are comparable, and completes the proof of the lemma. □

The next example shows that Lemma 8 does not admit a generalization to the case $S(r^1) = S(r^2) = \{\tau_1, \tau_2\} \neq \{\alpha, \beta\}$.

Example 9. *Let*

$$\Gamma = \mathbb{S}^1 \cup ([1, 2] \times \{0\}) \subseteq \mathbb{R}^2.$$

Then

$$r^1\colon [0, 2\pi + 1] \to \Gamma, \quad r^1(\tau) = \begin{cases} (\cos\tau, \sin\tau), & \tau \in [0, 2\pi\rangle \\ \tau - 2\pi + 1, & \tau \in [2\pi, 2\pi + 1] \end{cases}$$

$$r^2 \colon [0, 2\pi + 1] \to \Gamma, \quad r^2(\tau) = \begin{cases} (\cos\tau, -\sin\tau), & \tau \in [0, 2\pi\rangle \\ \tau - 2\pi + 1, & \tau \in [2\pi, 2\pi + 1] \end{cases},$$

is a pair of parametrizations of Γ*. Notice that* $S(r^1) = S(r^2) = \{0, 2\pi\} \neq \{0, 2\pi + 1\}$*. (Namely,* $r^i(0) = r^i(2\pi) = (1, 0)$*,* $i = 1, 2$*, and* r^1 *"winds up"* $[0, 2\pi] \subseteq [0, 2\pi+1]$ *onto* $\mathbb{S}^1$ *"counterclockwise - +", while* r^2 *- "winds up clockwise -".) Let*

$$\omega \colon [0, 2\pi + 1] \to [0, 2\pi + 1]$$

be a function such that

$$r^2(\omega|[0, 2\pi + 1] \setminus \{0, 2\pi, 2\pi + 1\}) = r^1|[0, 2\pi + 1] \setminus \{0, 2\pi, 2\pi + 1\}).$$

According to Remark 2 and the appropriate part of the proof of Lemma 8, the restriction $\omega|\,\langle 0, 2\pi\rangle$ *has to be strictly decreasing, while the restriction* $\omega|\,\langle 2\pi, 2\pi + 1\rangle$ *is obviously strictly increasing. Therefore,* ω *is neither piecewise strictly increasing nor piecewise strictly decreasing, so it is not a parametric substitute. Now, by Remark 5 (the uniqueness),* r^1 *and* r^2 *are* not *comparable.*

We also need a slight generalization of Lemma 8.

Lemma 10. *Given a* 1*-parametrizable set* $\Gamma \subseteq \mathbb{R}^n$*, let* $r^i \colon [\alpha_i, \beta_i] \to \Gamma$*,* $i = 1, 2$*, be a pair of its parametrizations. If* $S(r^1) \subseteq \{\alpha_1, \beta_1\}$ *and* $S(r^2) \subseteq \{\alpha_2, \beta_2\}$*, then* r^1 *and* r^2 *are comparable.*

Proof. Let

$$w \colon [\alpha_1, \beta_1] \to [\alpha_2, \beta_2],\ w(\alpha_1) = \alpha_2,\ w(\beta_1) = \beta_2,$$

be the (unique) affine function. Then w is a strictly increasing homeomorphism. It is obvious that

$$r^2 w \colon [\alpha_1, \beta_1] \to \Gamma$$

is also a parametrization of Γ (It is a continuous surjection having the singular set $S(r^2 w)$ of the same cardinality as $S(r^2)$.). Further, the inverse

$$w^{-1} \colon [\alpha_2, \beta_2] \to [\alpha_1, \beta_1]$$

of w is a strictly increasing homeomorphism as well, and hence, a parametric substitute. Since $(r^2 w) w^{-1} = r^2$, it follows by definition that r^2 is comparable to $r^2 w$. Observe that

$$S(r^2 w) = w^{-1}[S(r^2)] \subseteq w^{-1}[\{\alpha_2, \beta_2\}] = \{\alpha_1, \beta_1\}.$$

By Lemma 8, the parametrizations $r^2 w$ and r^1 are comparable. The final conclusion follows by Lemma 6. $\square$

We can now state and prove the main theorem (compare [5, 4. Theorem 29.2; pp. 253–255]).

Theorem 11. *For every* 1*-parametrizable set* $\Gamma \subseteq \mathbb{R}^n$*, the following statements are mutually equivalent:*

(a) Every two parametrizations of Γ *are comparable.*

(b) For every parametrization $r\colon [\alpha, \beta] \to \Gamma$*, the singular set* $S(r) \subseteq \{\alpha, \beta\}$*.*

(c) There exists a parametrization $p\colon [\gamma, \delta] \to \Gamma$ *having the singular set* $S(p) \subseteq \{\gamma, \delta\}$*.*

Proof. It suffices to prove the equivalences (a) $\Leftrightarrow$ (b) and (b) $\Leftrightarrow$ (c). Since the implication (b) $\Rightarrow$ (a) holds by Lemma 10, , while (b) $\Rightarrow$ (c) holds trivially, it remains to prove the implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (b).
(a) $\Rightarrow$ (b). Let us assume to the contrary, i.e., let (a) hold and let there exist a parametrization $r\colon [\alpha, \beta] \to \Gamma$ such that $S(r) \supseteq \{\tau_1, \tau_2\} \not\subseteq \{\alpha, \beta\}$ and $r(\tau_1) = r(\tau_2)$. Then $\alpha < \tau_1 < \tau_2 \leq \beta$ or $\alpha \leq \tau_1 < \tau_2 < \beta$. Consider the function

$$\omega\colon [\alpha, \beta] \to [\alpha, \beta], \quad w(\tau) = \begin{cases} \tau, & \alpha \leq \tau \leq \tau_1 \\ \tau_1 + \tau_2 - \tau, & \tau_1 < \tau < \tau_2 \\ \tau, & \tau_2 \leq \tau \leq \beta \end{cases}.$$

Notice that ω is a piecewise strictly monotone bijection, which is neither piecewise strictly increasing nor piecewise strictly decreasing. Therefore, ω is not a parametric substitute. Let

$$p\colon [\alpha, \beta] \to \Gamma, \quad p(\tau) = \begin{cases} r(\tau), & \alpha \leq \tau \leq \tau_1 \\ r(\tau_1 + \tau_2 - \tau), & \tau_1 < \tau < \tau_2 \\ r(\tau), & \tau_2 \leq \tau \leq \beta \end{cases}.$$

Since $r(\tau_1) = r(\tau_2)$, the function p is continuous, and it is a surjection. Further, $p = r\omega$ obviously holds, and since ω is bijective, $S(p) = \omega[S(r)]$, and thus, $|S(p) = ||S(r)|$. Therefore, p is a parametrization of Γ. However, since ω is not a parametric substitute, it follows (see Remark 5) that p and r are not mutually comparable parametrizations of Γ — a contradiction.
(c) $\Rightarrow$ (b). Let there exist a parametrization $p\colon [\gamma, \delta] \to \Gamma$ having the singular set $S(p) \subseteq \{\gamma, \delta\}$. Then, either $S(p) = \emptyset$ or $S(p) = \{\gamma, \delta\}$ (see the beginning of the proof of Lemma 8), We are to prove that, for every parametrization $r\colon [\alpha, \beta] \to \Gamma$, either $S(r) = \emptyset$ (whenever $S(p) = \emptyset$) or $S(r) = \{\alpha, \beta\}$ (whenever $S(p) = \{\gamma, \delta\}$) holds true. Let, firstly,

$S(p) = \emptyset$. Then p is a continuous bijection, and since Γ is a compactum, p is a homeomorphism. Thus, $\Gamma \approx [\alpha, \beta]$. Assume to the contrary, i.e., let there exist a parametrization $r\colon [\alpha, \beta] \to \Gamma$ having the singular set $S(r) \neq \emptyset$. Then there are $\tau_1, \tau_2 \in S(r)$, $\tau_1 < \tau_2$, such that

$$r| \langle \tau_1, \tau_2 \rangle : \langle \tau_1, \tau_2 \rangle \to r[\langle \tau_1, \tau_2 \rangle] \subseteq \Gamma$$

is a homeomorphism, and $\mathbb{S}^1 \approx r[[\tau_1, \tau_2]] \equiv \Gamma_0 \subseteq \Gamma \approx [\alpha, \beta]$.
It follows that the segment $[\alpha, \beta]$ contains a subspace which is homeomorphic to the circle $\mathbb{S}^1$. Consequently, $\mathbb{S}^1$ admits a continuous injection to $[\alpha, \beta]$ - a contradiction. Let $S(p) = \{\gamma, \delta\}$. Then $\Gamma \approx \mathbb{S}^1$. Assume again to the contrary, i.e., let there exist a parametrization $r\colon [\alpha, \beta] \to \Gamma$ having the singular set $S(r) \neq \{\alpha, \beta\}$. Then either $S(r) = \emptyset$ or $S(r) \supseteq \{\tau_1, \tau_2\} \neq \{\alpha, \beta\}$. By the previously proven case, $S(r) = \emptyset$ implies $S(p) = \emptyset$ — a contradiction. It remains the subcase $S(r) \supseteq \{\tau_1, \tau_2\} \neq \{\alpha, \beta\}$. We may assume, without loss of generality that $\tau_1 < \tau_2$ and there is no singular point between them. Then

$$r| \langle \tau_1, \tau_2 \rangle : \langle \tau_1, \tau_2 \rangle \to r[\langle \tau_1, \tau_2 \rangle] \subseteq \Gamma$$

is a homeomorphism, and thus, $r[[\tau_1, \tau_2]] \equiv \Gamma_0 \approx \mathbb{S}^1$. Since $\{\tau_1, \tau_2\} \neq \{\alpha, \beta\}$, it follows that $r[\langle \alpha, \tau_1 \rangle]$ or $r[\langle \tau_2, \beta \rangle]$ is a non-empty subset of $\Gamma \backslash \Gamma_0$. (If it is not so, the singular set $S(r)$ must be infinite, and r cannot be a parametrization of Γ.) Therefore, $\Gamma_0 \subsetneqq \Gamma$ holds, and consequently,

$$\mathbb{S}^1 \approx \Gamma_0 \subsetneqq \Gamma \approx \mathbb{S}^1$$

— a contradiction (Every *non-trivial* subset A of $\mathbb{S}^1$ is not homeomorphic to $\mathbb{S}^1$. Indeed, if A is connected, then it is an arc on $\mathbb{S}^1$ that is homeomorphic to a segment or an interval, while, if A is not connected, then it cannot be homeomorphic to the connected $\mathbb{S}^1$.) $\square$

By Theorem 11, the next definition is correct (compare [5, 4. Definitions 29.4 and 29.5; pp. 255–256]).

Definition 12. *A 1-parametrizable set $\Gamma \subseteq \mathbb{R}^n$ that admits a parametrization r having the singular set $S(r)$ empty is said to be a **simple curve** (with the boundary) or an **arc**. In that case, $r(\alpha), r(\beta) \in \Gamma$ are said to be the **boundary** or **end points** of Γ. If Γ admits a parametrization $r\colon [\alpha, \beta] \to \Gamma$ having the singular set $S(r) = \{\alpha, \beta\}$, then Γ is said to be a **simply closed curve**.*

Corollary 13. *Let $\Gamma \subseteq \mathbb{R}^n$ be a subset.*

(i) Γ is a simple curve, if and only if there exists a continuous bijection $r\colon [\alpha, \beta] \to \Gamma$;

(ii) Γ *is a simply closed curve, if and only if there exists a continuous surjection* $r\colon [\alpha,\beta] \to \Gamma$ *having the restriction* $r|\,\langle\alpha,\beta\rangle : \langle\alpha,\beta\rangle \to \Gamma$ *injective and* $r(\alpha) = r(\beta)$.

Corollary 14. *Every* 1*-parametrizable set* $\Gamma \subseteq \mathbb{R}^n$ *admits at most finitely many curves* $(\Gamma, [r])$.

Proof. Since, by definition, every 1-parametrizable set $\Gamma \subseteq \mathbb{R}^n$ is a "very specific" (compare Example 1 above) union of finitely many sets homeomorphic to a segment (simple curves) and finitely many sets homeomorphic to a circle (simply closed curves), the conclusion follows straightforwardly by Theorem 11, Definitions 7 and 17 and Corollary 13. □

Example 15.

(a) *Let* $\Gamma \subseteq \mathbb{R}^n$ *be a circle with an attached segment (see the last sentence of Example 1 and Example 9), or simply,*

$$\Gamma = \mathbb{S}^1 \sqcup_A \overline{AB} \subseteq \mathbb{R}^2,\ A \in \mathbb{S}^1.$$

Then, for every parametrization $r\colon [\alpha,\beta] \to \Gamma$, *either* $r(\alpha) = A$ *and* $r(\beta) = B$, *or* $r(\alpha) = B$ *and* $r(\beta) = A$. *One straightforwardly verifies that in each case there are exactly two comparability classes on* Γ. *Since each class of the first case equals to a unique (the "opposite" parametrization) class of the second case, it follows that* Γ *carries exactly two curves.*

(b) *Let* $\Gamma \subseteq \mathbb{R}^n$ *be a circle with two attached segments (at the same point — see the last sentence of Example 1), or simply,*

$$\Gamma = \mathbb{S}^1 \sqcup_A (\overline{AB} \sqcup_A \overline{AC}) \subseteq \mathbb{R}^2, A \in \mathbb{S}^1.$$

Then, for every parametrization $r\colon [\alpha,\beta] \to \Gamma$, *either* $r(\alpha) = B$ *and* $r(\beta) = C$, *or* $r(\alpha) = C$ *and* $r(\beta) = B$. *One straightforwardly verifies that in each case there are exactly two comparability classes on* Γ. *Since each class of the first case equals to a unique class of the second case, it follows that* Γ *carries exactly two curves.*

(c) *Let* $\Gamma \subseteq \mathbb{R}^n$ *be a "figure-8" set, or simply,*

$$\Gamma = X \sqcup_{(0,0)} Y \subseteq \mathbb{R}^2,$$

where

$$X = \{(\xi,\eta) \mid (\xi+1)^2 + \eta^2 = 1\},$$

$$Y = \{(\xi,\eta) \mid (\xi-1)^2 + \eta^2 = 1\}.$$

Then, for every parametrization $r\colon [\alpha,\beta] \to \Gamma$*, either* $r(\alpha) = r(\beta) = (\xi_0,\eta_0) \neq (0,0)$ *or* $r(\alpha) = r(\beta) = (0,0)$*. In the first case,* $S(r) = \{\alpha,\tau_1,\tau_2,\beta\}$ *such that* $r(\alpha) = r(\beta) = (\xi_0,\eta_0) \neq (0,0) = r(\tau_1) = r(\tau_2)$*, while in the second case,* $S(r) = \{\alpha,\sigma_1,\beta\}$ *such that* $r(\alpha) = r(\sigma_1) = r(\beta) = (0,0)$*. One easily verifies that in each case there are exactly two comparability classes on* Γ*. A little more careful examination shows that, in the first case, the classes do not depend on the point* $r(\alpha) = r(\beta) \in X \setminus \{(0,0)\}$ *nor on* $r(\alpha) = r(\beta) \in Y \setminus \{(0,0)\}$*. Then, further, one easily concludes that the classes do not depend on* $r(\alpha) = r(\beta) \in \Gamma$ *at all. Finally, one straightforwardly shows that each class of the first case equals to a unique class of the second case. Therefore, a "figure-8" set carries exactly two curves.*

The following example generalizes those given in Example 15.

Example 16.

(i) Let $\Gamma \subseteq \mathbb{R}^n$ *be an arc with* m *circles attached, each at different point. Then the following three (mutually non-homeomorphic) types can occur:*

(a) no attaching point is an end point of the arc;

(b) only one of the attaching points is an end point of the arc;

(c) exactly two of the attaching points are the end points of the arc.

Type (a) is equivalent to the segment $[0,3m{-}1] \subseteq \mathbb{R}$ *with the circles* $\{(\xi,\eta) \mid (\xi{-}(3j{-}2))^2{+}(\eta{+}1)^2 = 1\} \subseteq \mathbb{R}^2$*,* $j = 1,\dots,m$*, considered as the subspace of* $\mathbb{R}^2$*. Observe that, for every parametrization* $r\colon [\alpha,\beta] \to \Gamma$*,*

$$S(r) = \{\tau_1,\tau_2,\dots,\tau_{2m-1},\tau_{2m}\}$$

such that either $r(\alpha) = (0,0)$*,* $r(\tau_{2j-1}) = r(\tau_{2j}) = (0,3j-2)$*,* $j = 1,\dots,m$*, and* $r(\beta) = (0,3m-1)$*, or* $r(\alpha) = (3m-1,0)$*,* $r(\tau_{2j-1}) = r(\tau_{2j}) = (0,3(m-j)+1)$*,* $j = 1,\dots,m$*, and* $r(\beta) = (0,0)$*. In each case there are exactly* $1{\cdot}2{\cdot}\cdots{\cdot}2{\cdot}1 = 2^m$ *comparability classes on* Γ*. Since each class of the first case equals to a unique (the "opposite" parametrization) class of the second case, it follows that* Γ *carries exactly* 2^m *curves. By a similar examination, one easily establishes the same number of curves for types (b) and (c) as well.*

(ii) Let $\Gamma \subseteq \mathbb{R}^n$ *be an* m-bouquet *(finite,* $m \in \mathbb{N}$*) of circles, or simply,*

$$\Gamma = \cup_{j=1}^{m} S_j^1 \subseteq \mathbb{R}^2, \quad S_j^1 \equiv \left\{ (\xi, \eta) \in \mathbb{R}^2 \;\middle|\; \xi^2 + (\eta + \frac{1}{j})^2 = \frac{1}{j^2} \right\}.$$

Then, for every parametrization $r\colon [\alpha, \beta] \to \Gamma$,

$$S(r) = \{\alpha, \tau_1, \tau_2, \dots, \tau_{2m-3}, \tau_{2m-2}, \beta\}$$

such that $r(\alpha) = r(\beta)$ *and* $r(\tau_{2j-1}) = r(\tau_{2j}) = (0,0)$, $j = 1, \dots, m-1$. *Further, with respect to the comparability,* $r(\alpha) = r(\beta) \in \Gamma$ *may be any point (see Example 15 (c)). Now, a straightforward analysis shows that* Γ *carries exactly* $1 \cdot 2 \cdot \dots \cdot 2 = 2^{m-1}$ *curves.*

3.2 The oriented curve

We have defined a curve to be an ordered pair $(\Gamma, [r])$ consisting of a 1-parametrizable set $\Gamma \subseteq \mathbb{R}^n$ and the comparability (equivalence) class $[r]$ of a parametrization r of Γ. Hereby $r^1, r^2 \in [r]$ if and only if there exists a parametric substitute, that is an either piecewise strictly increasing or piecewise strictly decreasing function w, such that $r^2 w = r^1$. We shall now separate those two possibilities (compare [5, 4. Definition 29.6; p. 256]).

Definition 17. *A parametrization* $r^2\colon [\alpha_2, \beta_2] \to \Gamma$ *is said to be* ***coherent*** *with a parametrization* $r^1\colon [\alpha_1, \beta_1] \to \Gamma$*, if there exists a piecewise strictly increasing function* $w\colon [\alpha_1, \beta_1] \to [\alpha_2, \beta_2]$ *such that* $r^2 w = r^1$.

Lemma 18. *Given a* 1*-parametrizable set* $\Gamma \subseteq \mathbb{R}^n$*, the coherency is an equivalence relation on the set of all parametrizations* r *of* Γ*. The coherency (equivalence) class of an* r *is denoted by* $\lceil r \rceil$.

Proof. Lemma can be proven in the same way as Lemma 6, so we omit the explicit proof. □

Theorem 19. *Let* $\Gamma \subseteq \mathbb{R}^n$ *be a* 1*-parametrizable set. Then,*

(i) For every pair r^1, r^2 *of parametrizations of* Γ*, if* r^2 *is coherent with* r^1*, then* r^2 *is comparable to* r^1*, i.e., for every parametrization* r *of* Γ, $\lceil r \rceil \subseteq [r]$ *holds.*

(ii) Each comparability class $[r]$ *splits into exactly two coherency classes* $\lceil r \rceil$ *and* $\lceil p \rceil$*, i.e., for every parametrization* r *of* Γ*, there exists a* $p \in [r]$ *such that* $[r] = \lceil r \rceil \cup \lceil p \rceil$ *and* $\lceil r \rceil \cap \lceil p \rceil = \emptyset$.

Proof. Statement (i) follows by Definitions 4 and 17, while statement (ii) follows by (i) and the same definitions. □

Definition 20. *An* ***oriented curve*** *is an ordered pair* $(\Gamma, \lceil r\rceil)$ *consisting of a* 1*-parametrizable set* $\Gamma \subseteq \mathbb{R}^n$ *and the coherency class* $\lceil r\rceil$ *of a parametrization* r *of* Γ. *The coherency class* $\lceil r\rceil$ *is said to be the* ***orientation*** *of* $(\Gamma, \lceil r\rceil)$. *Hereby,* $x_1 \equiv r(\alpha) \in \Gamma$ *is said to be the* ***starting point****, while* $x_2 \equiv r(\beta) \in \Gamma$ *is said to be the* ***ending point*** *of the oriented curve* $(\Gamma, \lceil r\rceil)$.

Observe that, by Theorem 19 (ii), each curve $(\Gamma, [r])$ admits exactly two orientations, i.e., $(\Gamma, [r])$ "carries" exactly two oriented curves $(\Gamma, \lceil r\rceil)$ and $(\Gamma, \lceil p\rceil)$, $[p] = [r] = \lceil r\rceil \cup \lceil p\rceil$, $\lceil r\rceil \cap \lceil p\rceil = \varnothing$. Then one usually says that a parametrization $r^1 \in \lceil r\rceil$ and a parametrization $p^1 \in \lceil p\rceil$ are mutually *opposite*, and the brief notations $\overset{\curvearrowright}{\Gamma}$ and $\overset{\curvearrowleft}{\Gamma}$ are used as well. As an immediate consequence (see also Corollary 2), for instance, every "figure 8" subspace of $\mathbb{R}^n$, $n \geq 2$, is a 1-parametrizable set carrying exactly two curves and exactly four oriented curves. Notice that, for a continuously differentiable parametrization r, the orientation $\lceil r\rceil$ is determined by $sgn(w')$. Further, observe that if $(\Gamma, [r])$ is a simply closed curve, then $r(\beta) = r(\alpha)$, for every parametrization r. So the "starting" = "ending" point makes sense for each parametrized set (Γ, r) only.

3.3 About invariants

Concerning the well known invariants of curves (the flexion and torsion at a sufficiently smooth point, the length when a parametrization has a bounded variation, the line integrals when a parametrization is piecewise differentiable, ...), nothing essentially changes (see (a), (b), (c), (d) and (e) below). We shall hereby, for instance, consider the length of a curve only. Firstly, recall the notion of a variation of a function. Let $T \subseteq \mathbb{R}$ denote any of $\langle\alpha, \beta\rangle$, $\langle\alpha, \beta]$, $[\alpha, \beta\rangle$ or $[\alpha, \beta]$ in $\mathbb{R}$, and let

$$D = \{\tau_0, \tau_1, \ldots, \tau_l\}, \quad \tau_0 < \tau_1 < \cdots < \tau_l,$$

be any partition of T such that $\tau_0 = \alpha$ whenever either $T = [\alpha, \beta\rangle$ or $T = [\alpha, \beta]$, while $\tau_l = \beta$ whenever either $T = \langle\alpha, \beta]$ or $T = [\alpha, \beta]$. It is obvious that the set $\mathcal{D}(T)$ of all partitions D of T is partially ordered by inclusion. Given a function $f\colon T \to \mathbb{R}^n$, the *variation of f with respect to D* is defined by

$$V(f; D) \equiv \sum_{k=1}^{l} \|f(\tau_k) - f(\tau_{k-1})\| \in \{0\} \cup \mathbb{R}^+.$$

Clearly, if $D' \supseteq D$, then $V(f; D') \geq V(f; D)$. If the set

$$\{V(f; D) \mid D \in \mathcal{D}(T)\} \subseteq \{0\} \cup \mathbb{R}^+$$

is bounded, then f is said to be a function *of bounded variation*. In that case, the (unique) real number

$$V(f) \equiv \sup\{V(f,D) \mid D \in \mathcal{D}(T)\} \in \{0\} \cup \mathbb{R}^+$$

is said to be the (*total*) *variation of* f.

Example 21. *One straightforwardly shows that the well known Koch curve (either simple - homeomorphic to a segment, or simply closed - homeomorphic to a circle) does not admit any parametrization being a function of bounded variation. Further, the continuously differentiable function*

$$g\colon T_1 = \langle 0,1] \to \mathbb{R}, \quad g(\tau) = \tau \cos \frac{2\pi}{\tau},$$

is not a function of bounded variation. (Notice that T_1 *is not a segment.) Similarly, the function*

$$h\colon T_2 = [0,1] \to \mathbb{R}, \quad h(\tau) = \begin{cases} \tau \cos \frac{2\pi}{\tau}, & \tau \neq 0 \\ 0, & \tau = 0 \end{cases}$$

(the trivial extension of f *to* $[0,1]$*) is not a function of bounded variation.*

The following facts of real analysis are well known:

(a) A (vectorial) function $f = (f_1, \dots, f_n)\colon T \to \mathbb{R}^n$ is a function of bounded variation, if and only if every (scalar) function $f_j\colon T \to \mathbb{R}$, $j = 1, \dots, n$, is a function of bounded variation.

(b) Every monotone function $f\colon T \to \mathbb{R}$ is a function of bounded variation.

(c) Every continuously differentiable function $f\colon T = [\alpha, \beta] \to \mathbb{R}^n$ is a function of bounded variation.

(d) Every piecewise continuously differentiable function $f\colon T = [\alpha, \beta] \to \mathbb{R}^n$ is a function of bounded variation. (Hereby "piecewise" means that there exists a partition $D = \{\tau_0, \tau_1, \dots, \tau_l\} \in \mathcal{D}(T)$ such that each restriction $f|[\tau_{k-1}, \tau_k]$, $k = 1, \dots, l$, is continuously differentiable.)

(e) Let, for a given pair $f_i\colon T_i \to \mathbb{R}^n$, $i = 1, 2$, there exist a monotone bijection $w\colon T_1 \to T_2$ such that $f_1 = f_2 w$. Then f_2 is a function of bounded variation, if and only if f_1 is a function of bounded variation, and in that case $V(f_2) = V(f_1)$.

The facts from above admit a correct definition of the length of a 1-parametrized set as well as the length of a curve.

Definition 22. *A* 1*-parametrized set* (Γ, r) *is said to* ***have a length*** *(or that it is* ***rectifiable****), if the parametrization* $r\colon [\alpha, \beta] \to \Gamma$ *is a function of bounded variation. In that case, the (total) variation* $V(r)$ *is said to be the* ***length*** *of* (Γ, r)*, usually denoted by* $L(\Gamma, r)$*.*

The mentioned facts readily imply the following:

(i) Let 1-parametrized sets (Γ, r^i) have lengths $L(\Gamma, r^i)$, $i = 1, 2$. If there exists a piecewise monotone function $\omega\colon [\alpha_1, \beta_1] \to [\alpha_2, \beta_2]$ such that $r^2\omega = r^1$, then $L(\Gamma, r^2) = L(\Gamma, r^1)$.

(ii) Let $r^i\colon [\alpha_i, \beta_i] \to \Gamma$, $i = 1, 2$, be a pair of parametrization of a 1-parametrizable set $\Gamma \subseteq \mathbb{R}^n$. If r^1 and r^2 are mutually comparable and they both are functions of bounded variation, then $L(\Gamma, r^2) = L(\Gamma, r^1)$.

Consequently, the property **to have a length** and, in that case, the notion of a **length** for a curve $(\Gamma, [r])$ can be correctly defined by $V(r)$ via any parametrization $r\colon [\alpha, \beta] \to \Gamma$.

Remark 23. *The question arises, whether this approach to the notion of a curve admits a generalization in order to define the notion of a surface in* $\mathbb{R}^n$*. It seems to be possible in several special cases (a simple surface, sphere, cylinder, Möbius band, torus, Klein bottle). However, in general, given an* $\Sigma \subseteq \mathbb{R}^n$ *admitting a surjective mapping*

$$r\colon [\alpha, \beta] \times [\gamma, \delta] \to \Sigma,$$

we do not know all needed conditions on the singular set $S(r)$ *making* Σ *"2-parametrizable".*

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