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# Nonelementary irreducible representations of $\mathrm{Spin}(n,1)$

Domagoj Kovačević, Hrvoje Kraljević

**Abstract**

We study corners and fundamental corners of the irreducible subquotients of reducible elementary representations of the groups $G = \mathrm{Spin}(n,1)$. For even $n$ we obtain results in a way analogous to the results in [8] for the groups $\mathrm{SU}(n,1)$. Especially, we again get a bijection between the nonelementary part $\hat{G}^0$ of the unitary dual $\hat{G}$ and the unitary dual $\hat{K}$. In the case of odd $n$ we get a bijection between $\hat{G}^0$ and a true subset of $\hat{K}$.

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# 1 Introduction

## 1.1 Elementary representations

Let $G$ be a connected semisimple Lie group with finite center, $\mathfrak{g}_0$ its Lie algebra, $K$ its maximal compact subgroup, and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ the corresponding Cartan decomposition of $\mathfrak{g}_0$. Let $\mathfrak{a}_0$ be a Cartan subspace of $\mathfrak{p}_0$, $A$ the corresponding vector subgroup of $G$ and $M$ (resp. $\mathfrak{m}_0$) the centralizer of $A$ in $K$ (resp. of $\mathfrak{a}_0$ in $\mathfrak{k}_0$). Let $P = MAN$ be the minimal parabolic subgroup of $G$ corresponding to a choice of positive restricted roots of the pair $(\mathfrak{g}_0, \mathfrak{a}_0)$. For any compact group $L$ its unitary dual will be denoted by $\hat{L}$. Furthermore, we denote by $\mathfrak{l}$ the complexification of a real vector space $\mathfrak{l}_0$. For $\sigma \in \hat{M}$ and $\nu \in \mathfrak{a}^*$ let $\pi^{\sigma,\nu}$ be the corresponding elementary representation of $G$ – the representation parabolically induced by the representation $\sigma \otimes \nu$ of $P$.

From classical results of Harish–Chandra we know that all elementary representations are admissible and of finite length and that every completely irreducible admissible representation of $G$ on a Banach space is infinitesimally equivalent to an irreducible subquotient of an elementary representation. Infinitesimal equivalence of completely irreducible admissible representations is equivalent to algebraic equivalence of the corresponding $(\mathfrak{g}, K)$–modules. We will denote by $\widehat{G}$ the set of all infinitesimal equivalence classes of completely irreducible admissible representations of $G$ on Banach spaces. $\widehat{G}^e$ will denote the set of infinitesimal equivalence classes of irreducible elementary representations and $\widehat{G}^0 = \widehat{G} \setminus \widehat{G}^e$ the set of infinitesimal equivalence classes of irreducible suquotients of reducible elementary representations. It is also due to Harish–Chandra that every irreducible unitary representation is admissible and that infinitesimal equivalence between such representations is equivalent to their unitary equivalence. Thus the unitary dual $\hat{G}$ of $G$ can be regarded as a subset of $\widehat{G}$. We denote $\hat{G}^e = \hat{G} \cap \widehat{G}^e$ and $\hat{G}^0 = \hat{G} \cap \widehat{G}^0 = \hat{G} \setminus \hat{G}^e$.

## 1.2 Infinitesimal characters

We denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and by $\mathfrak{Z}(\mathfrak{g})$ its center. We denote by $\hat{\mathfrak{Z}}(\mathfrak{g})$ the set of all infinitesimal characters (unital homomorphisms $\mathfrak{Z}(\mathfrak{g}) \longrightarrow \mathbb{C}$) of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subseteq \mathfrak{h}^*$ the root system of the pair $(\mathfrak{g}, \mathfrak{h})$ and $W = W(\mathfrak{g}, \mathfrak{h})$ its Weyl group. Denote by $\mathcal{P}(\mathfrak{h}^*)$ the polynomial algebra over $\mathfrak{h}^*$ and by $\omega = \omega_{\mathfrak{h}}$ the Harish–Chandra isomorphism of $\mathfrak{Z}(\mathfrak{g})$ onto the algebra $\mathcal{P}(\mathfrak{h}^*)^W$ of $W$–invariant polynomials on $\mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$ define $\chi_\lambda \in \hat{\mathfrak{Z}}(\mathfrak{g})$ by $\chi_\lambda(z) = (\omega(z))(\lambda)$, $z \in \mathfrak{Z}(\mathfrak{g})$. Then $\lambda \mapsto \chi_\lambda$ is a surjection of $\mathfrak{h}^*$ onto $\hat{\mathfrak{Z}}(\mathfrak{g})$ and for $\lambda, \mu \in \mathfrak{h}^*$ one has $\chi_\lambda = \chi_\mu$ if and only if $\mu = w\lambda$ for some $w \in W$.

It is well known that every elementary representation $\pi^{\sigma,\lambda}$ has infinitesimal character. To describe it chose a Cartan subalgebra $\mathfrak{d}_0$ of $\mathfrak{m}_0$ and let $\Delta_{\mathfrak{m}}^+$ be a choice of positive roots of the pair $(\mathfrak{m}, \mathfrak{d})$. Set $\delta_{\mathfrak{m}} = \frac{1}{2} \sum_{\alpha \in \Delta_{\mathfrak{m}}^+} \alpha$. Denote by $\lambda_\sigma \in \mathfrak{d}^*$ the highest weight of the representation $\sigma$ with respect to $\Delta_{\mathfrak{m}}^+$. Now, $\mathfrak{h}_0 = \mathfrak{d}_0 \dotplus \mathfrak{a}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ and the infinitesimal character of the elementary representation $\pi^{\sigma,\nu}$ is $\chi_{\Lambda(\sigma,\nu)}$, where $\Lambda(\sigma,\nu)\in\mathfrak{h}^*$ is given by

$$\Lambda(\sigma,\nu)|_{\mathfrak{d}}=\lambda_\sigma+\delta_{\mathfrak{m}}\qquad\text{and}\qquad\Lambda(\sigma,\nu)|_{\mathfrak{a}}=\nu.$$

## 1.3 Corners and fundamental corners

Suppose now that the rank of $\mathfrak{g}$ is equal to the rank of $\mathfrak{k}$. Choose a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$. Let $\Delta_K=\Delta(\mathfrak{k},\mathfrak{t})\subseteq\Delta=\Delta(\mathfrak{g},\mathfrak{t})$ be the root systems of the pairs $(\mathfrak{k},\mathfrak{t})$ and $(\mathfrak{g},\mathfrak{t})$ and $W_K=W(\mathfrak{k},\mathfrak{t})\subseteq W=W(\mathfrak{g},\mathfrak{t})$ the corresponding Weyl groups. Choose positive roots $\Delta_K^+$ in $\Delta_K$ and let $C$ be the corresponding $W_K-$Weyl chamber in $\mathfrak{t}_{\mathbb{R}}^*=i\mathfrak{t}_0^*$. Denote by $\mathcal{D}$ the set of all $W-$Weyl chambers in $i\mathfrak{t}_0^*$ contained in $C$. For $D\in\mathcal{D}$ we denote by $\Delta^D$ the corresponding positive roots in $\Delta$ and let $\Delta_P^D$ denotes the noncompact roots in $\Delta^D$, i.e. $\Delta_P^D=\Delta^D\setminus\Delta_K^+$. Set

$$\rho_K=\frac{1}{2}\sum_{\alpha\in\Delta_K^+}\alpha\qquad\text{and}\qquad\rho_P^D=\frac{1}{2}\sum_{\alpha\in\Delta_P^D}\alpha.$$

Recall some definitions from [8]. For a representation $\pi$ of $G$ and for $q\in\hat{K}$ we denote by $(\pi:q)$ the multiplicity of $q$ in $\pi|K$. The $K-$**spectrum** $\Gamma(\pi)$ of a representation $\pi$ of $G$ is defined by

$$\Gamma(\pi)=\{q\in\hat{K};\ (\pi:q)>0\}.$$

We identify $q\in\hat{K}$ with its maximal weight in $i\mathfrak{t}_0^*$ with respect to $\Delta_K^+$. For $q\in\Gamma(\pi)$ and for $D\in\mathcal{D}$ we say:

(*i*) $q$ is a $D-$**corner** for $\pi$ if $q-\alpha\not\in\Gamma(\pi)\ \forall\alpha\in\Delta_P^D$;

(*ii*) $q$ is a $D-$**fundamental corner** for $\pi$ if it is a $D-$corner for $\pi$ and $\chi_{q+\rho_K-\rho_P^D}$ is the infinitesimal character of $\pi$;

(*iii*) $q$ is a **fundamental corner** for $\pi$ if it is a $D-$fundamental corner for $\pi$ for some $D\in\mathcal{D}$.

In [8] for the case of the groups $G=SU(n,1)$ and $K=U(n)$ the following results were proved:

1. Every $\pi\in\widehat{G}^0$ has either one or two fundamental corners.
2. $\hat{G}^0=\{\pi\in\widehat{G}^0;\ \pi$ has exactly one fundamental corner$\}$.
3. For $\pi\in\hat{G}^0$ denote by $q(\pi)$ the unique fundamental corner of $\pi$. Then $\pi\mapsto q(\pi)$ is a bijection of $\hat{G}^0$ onto $\hat{K}$.

In this paper we investigate the analogous notions for the groups $\mathrm{Spin}(n,1)$.

## 2 The groups $\mathrm{Spin}(n,1)$

In the rest of the paper $G = \mathrm{Spin}(n,1)$, $n \geq 3$, is the connected and simply connected real Lie group with simple real Lie algebra

$$\mathfrak{g}_0 = \mathfrak{so}(n,1) = \left\{A \in \mathfrak{gl}(n+1,\mathbb{R});\ A^t = -\Gamma A \Gamma\right\}, \qquad \Gamma = \left[\begin{array}{cc} I_n & 0 \\ 0 & -1 \end{array}\right],$$

i.e.

$$\mathfrak{g}_0 = \left\{\left[\begin{array}{cc} B & a \\ a^t & 0 \end{array}\right];\ B \in \mathfrak{so}(n),\ a \in M_{n,1}(\mathbb{R})\right\}.$$

Here and in the rest of the paper we use the usual notation:

- $M_{m,n}(K)$ is the vector space of $m \times n$ matrices over a field $K$.
- $\mathfrak{gl}(n,K)$ denotes the Lie algebra $M_{n,n}(K)$ with $[A,B] = AB - BA$.
- $\mathrm{GL}(n,K)$ is the group of invertible matrices in $M_{n,n}(K)$.
- $A^t$ is the transpose of a matrix $A$.
- $\mathfrak{so}(n,K) = \{B \in \mathfrak{gl}(n,K);\ B^t = -B\}$.
- $\mathfrak{so}(n) = \mathfrak{so}(n,\mathbb{R})$.
- $\mathrm{SO}(n) = \{A \in \mathrm{GL}(n,\mathbb{R});\ A^{-1} = A^t,\ \det A = 1\}$.

We choose Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ :

$$\mathfrak{k}_0 = \left\{\left[\begin{array}{cc} B & 0 \\ 0 & 0 \end{array}\right];\ B \in \mathfrak{so}(n)\right\}, \qquad \mathfrak{p}_0 = \left\{\left[\begin{array}{cc} 0 & a \\ a^t & 0 \end{array}\right];\ a \in M_{n,1}(\mathbb{R})\right\}.$$

The group $\mathrm{Spin}(n,1)$ is double cover of the identity component $\mathrm{SO}(n,1)_0$ of the Lie group $\mathrm{SO}(n,1) = \{A \in \mathrm{GL}(n+1,\mathbb{R});\ A^{-1} = \Gamma A^t \Gamma,\ \det A = 1\}$. The maximal compact subgroup $K \subset G$ with Lie algebra $\mathfrak{k}_0$ is the double cover $\mathrm{Spin}(n)$ of the group $\mathrm{SO}(n)$.

Now we choose Cartan subalgebras. $E_{p,q}$ will denote the $(n+1)\times(n+1)$ matrix with $(p,q)-$entry equal 1 and all the other entries 0. Set

$$\begin{aligned} I_{p,q} &= E_{p,q} - E_{q,p}, \qquad && 1 \leq p,q \leq n, \qquad p \neq q; \\ B_p &= E_{p,n+1} + E_{n+1,p}, \qquad && 1 \leq p \leq n. \end{aligned}$$

Then $\{I_{p,q};\ 1 \leq q < p \leq n\}$ is a basis of $\mathfrak{k}_0$, $\{B_p;\ 1 \leq p \leq n\}$ is a basis of $\mathfrak{p}_0$ and $\mathfrak{t}_0 = \mathrm{span}_{\mathbb{R}}\left\{I_{2p,2p-1};\ 1 \leq p \leq \frac{n}{2}\right\}$ is a Cartan subalgebra of $\mathfrak{k}_0$.

We consider separately two cases: $n$ even and $n$ odd.

$n$ **even,** $n = 2k$

In this case $\mathfrak{t}_0$ is also a Cartan subalgebra of $\mathfrak{g}_0$. Set

$$H_p = -iI_{2p,2p-1}, \qquad 1 \le p \le k.$$

Dual space $\mathfrak{t}^*$ identifies with $\mathbb{C}^k$ through this basis of $\mathfrak{t}$ :

$$\mathfrak{t}^* \ni \lambda = (\lambda(H_1), \dots, \lambda(H_k)) \in \mathbb{C}^k.$$

Denoting by $\{\alpha_1, \dots, \alpha_k\}$ the canonical basis of $\mathbb{C}^k = \mathfrak{t}^*$ the root system of the pair $(\mathfrak{g}, \mathfrak{t})$ is

$$\Delta = \Delta(\mathfrak{g}, \mathfrak{t}) = \{\pm\alpha_p \pm \alpha_q;\ 1 \le p, q \le k,\ p \ne q\} \cup \{\pm\alpha_p;\ 1 \le p \le k\}.$$

The Weyl group $W$ of $\Delta$ consists of all permutations of the coordinates combined with multiplications of some coordinates with $-1$.

The root system $\Delta_K$ of the pair $(\mathfrak{k}, \mathfrak{t})$ is $\{\pm\alpha_p \pm \alpha_q;\ p \ne q\}$. We choose positive roots $\Delta_K^+ = \{\alpha_p \pm \alpha_q;\ 1 \le p < q \le k\}$. The unitary dual $\hat{K}$ of $K = \mathrm{Spin}(2k)$ will be parametrized by identifying with the corresponding highest weights. Thus

$$\hat{K} = \left\{(m_1, \dots, m_k) \in \mathbb{Z}^k \cup \left(\tfrac{1}{2} + \mathbb{Z}\right)^k;\ m_1 \ge m_2 \ge \cdots \ge m_{k-1} \ge |m_k|\right\}.$$

$n$ **odd,** $n = 2k+1$

Now $\mathfrak{t}_0$ is not a Cartan subalgebra of $\mathfrak{g}_0$. Set

$$H = B_n = B_{2k+1} = E_{2k+1,2k+2} + E_{2k+2,2k+1}, \quad \mathfrak{a}_0 = \mathbb{R}H, \quad \mathfrak{h}_0 = \mathfrak{t}_0 \dot{+} \mathfrak{a}_0.$$

Then $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ and all the other Cartan subalgebras of $\mathfrak{g}_0$ are $\mathrm{Int}(\mathfrak{g}_0)-$conjugated with $\mathfrak{h}_0$. The ordered basis $(H_1, \dots, H_k, H)$ of $\mathfrak{h}$ is used for the identification of $\mathfrak{h}^* = \mathbb{C}^{k+1}$ :

$$\mathfrak{h}^* \ni \lambda = (\lambda(H_1), \dots, \lambda(H_k), \lambda(H)) \in \mathbb{C}^{k+1}.$$

$\mathfrak{t}^*$ and $\mathfrak{a}^*$ are identified with subspaces of $\mathfrak{h}^*$ : $\mathfrak{t}^* = \{\lambda \in \mathfrak{h}^*;\ \lambda|_{\mathfrak{a}} = 0\}$ and $\mathfrak{a}^* = \{\lambda \in \mathfrak{h}^*;\ \lambda|_{\mathfrak{t}} = 0\}$. So $\mathfrak{h}^* = \mathfrak{t}^* \dot{+} \mathfrak{a}^*$.

Let $\{\alpha_1, \dots, \alpha_{k+1}\}$ be the canonical basis of $\mathbb{C}^{k+1}$. The root system of the pair $(\mathfrak{g}, \mathfrak{h})$ is $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) = \{\pm\alpha_p \pm \alpha_q;\ 1 \le p, q \le k+1,\ p \ne q\}$. The Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ consists of all permutations of coordinates combined with multiplying even number of coordinates with $-1$. The root system of the pair $(\mathfrak{k}, \mathfrak{t})$ is $\Delta_K = \Delta(\mathfrak{k}, \mathfrak{t}) = \{\pm\alpha_p \pm \alpha_q;\ 1 \le p, q \le k,\ p \ne q\} \cup \{\pm\alpha_p;\ 1 \le p \le k\}$. Choose positive roots $\Delta_K^+ = \{\alpha_p \pm \alpha_q;\ 1 \le p < q \le k\} \cup \{\alpha_p;\ 1 \le p \le k\}$. The dual $\hat{K}$ is again identified with the highest weights:

$$\hat{K} = \left\{q = (m_1, \dots, m_k) \in \mathbb{Z}_+^k \cup \left(\tfrac{1}{2} + \mathbb{Z}_+\right)^k;\ m_1 \ge m_2 \ge \cdots \ge m_k\right\}.$$

**Elementary representations of the groups** $\mathrm{Spin}(n,1)$

Regardless the parity of $n$ we put $H = B_n = E_{n,n+1} + E_{n+1,n}$, $\mathfrak{a}_0 = \mathbb{R}H$. As we already said, if $n$ is odd, $n = 2k+1$, then $\mathfrak{h}_0 = \mathfrak{t}_0 \dot{+} \mathfrak{a}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ and all the other Cartan subalgebras are $\mathrm{Int}(\mathfrak{g}_0)-$conjugated to $\mathfrak{h}_0$. If $n = 2k$ $\mathfrak{g}_0$ has two $\mathrm{Int}(\mathfrak{g}_0)-$conjugacy classes of Cartan subalgebras. Their representatives are $\mathfrak{t}_0$ and $\mathfrak{h}_0 = \mathrm{span}_{\mathbb{R}}\{iH_1, \ldots, iH_{k-1}, H\}$. $\mathfrak{t}$ and $\mathfrak{h}$ are of course $\mathrm{Int}(\mathfrak{g})-$conjugated. Explicitly, the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}}P_k & \frac{1}{\sqrt{2}}P_k & -ie_k \\ -\frac{1}{\sqrt{2}}Q_k & \frac{1}{\sqrt{2}}I_k & 0_k \\ -\frac{i}{\sqrt{2}}e_k^t & \frac{i}{\sqrt{2}}e_k^t & 0 \end{bmatrix} \in \mathrm{SO}(2k,1,\mathbb{C}),$$

where $P_k = I_k - E_{k,k} = \mathrm{diag}(1, \ldots, 1, 0)$,

$$Q_k = I_k - 2E_{k,k} = \mathrm{diag}(1, \ldots, 1, -1),$$

$e_k \in M_{k,1}(\mathbb{C})$ is given by $e_k^t = [0 \cdots 0\,1]$ and $0_k$ is the zero matrix in $M_{k,1}(\mathbb{C})$, has the properties $AH_jA^{-1} = H_j$, $1 \le j \le k-1$, and $AH_kA^{-1} = H$; thus, $A\mathfrak{t}A^{-1} = \mathfrak{h}$ and the parameters from $\mathbb{C}^k = \mathfrak{h}^* = \mathfrak{t}^*$ of the infinitesimal characters obtained through the two Harish$-$Chandra isomorphisms $\mathfrak{Z}(\mathfrak{g}) \longrightarrow \mathcal{P}(\mathfrak{h}^*)^W$ and $\mathfrak{Z}(\mathfrak{g}) \longrightarrow \mathcal{P}(\mathfrak{t}^*)^W$ coincide if the identifications of $\mathfrak{h}^*$ and $\mathfrak{t}^*$ with $\mathbb{C}^k$ are done through the two ordered bases $(H_1, \ldots, H_{k-1}, H)$ of $\mathfrak{h}$ and $(H_1, \ldots, H_{k-1}, H_k)$ of $\mathfrak{t}$.

For both cases, $n$ even and $n$ odd, $\mathfrak{m}_0$ is the subalgebra of all matrices in $\mathfrak{g}_0$ with the last two rows and columns 0. The subgroup $M$ is isomorphic to $\mathrm{Spin}(n-1)$. A Cartan subalgebra of $\mathfrak{m}_0$ is

$$\mathfrak{d}_0 = \mathfrak{t}_0 \cap \mathfrak{m}_0 = \mathrm{span}_{\mathbb{R}}\{iH_1, \ldots, iH_{k-1}\}, \qquad k = \left\lfloor \frac{n}{2} \right\rfloor.$$

The elements of $\hat{M}$ are identified with their highest weights. For $n$ even, $n = 2k$, we have

$$\hat{M} = \left\{(n_1, \ldots, n_{k-1}) \in \mathbb{Z}_+^{k-1} \cup \left(\tfrac{1}{2} + \mathbb{Z}_+\right)^{k-1};\ n_1 \ge n_2 \ge \cdots \ge n_{k-1} \ge 0\right\}$$

and for $n$ odd, $n = 2k+1$, we have

$$\hat{M} = \left\{(n_1, \ldots, n_k) \in \mathbb{Z}^k \cup \left(\tfrac{1}{2} + \mathbb{Z}\right)^k;\ n_1 \ge n_2 \ge \cdots \ge n_{k-1} \ge |n_k|\right\}.$$

The branching rules for the restriction of representations of $K$ to the subgroup $M$ are the following:

If $n$ is even, $n=2k$, we have

$$(m_1,\ldots,m_k)|_M = \bigoplus_{(n_1,\ldots,n_{k-1})\prec(m_1,\ldots,m_k)} (n_1,\ldots,n_{k-1});$$

here the symbol $(n_1,\ldots,n_{k-1}) \prec (m_1,\ldots,m_k)$ means that all $m_i$ and $n_j$ are either in $\mathbb{Z}$ or in $\frac{1}{2}+\mathbb{Z}$ and that

$$m_1 \geq n_1 \geq m_2 \geq n_2 \cdots \geq m_{k-1} \geq n_{k-1} \geq |m_k|.$$

If $n$ is odd, $n=2k+1$, we have

$$(m_1,\ldots,m_k)|_M = \bigoplus_{(n_1,\ldots,n_k)\prec(m_1,\ldots,m_k)} (n_1,\ldots,n_k);$$

now the symbol $(n_1,\ldots,n_k) \prec (m_1,\ldots,m_k)$ means that all $m_i$ and $n_j$ are either in $\mathbb{Z}$ or in $\frac{1}{2}+\mathbb{Z}$ and that

$$m_1 \geq n_1 \geq m_2 \geq n_2 \cdots \geq m_{k-1} \geq n_{k-1} \geq m_k \geq |n_k|.$$

The restriction $\pi^{\sigma,\nu}|_K$ is the representation of $K$ induced by the representation $\sigma$ of the subgroup $M$, thus it does not depend on $\nu$. By Frobenius Reciprocity Theorem the multiplicity of $q\in\hat{K}$ in $\pi^{\sigma,\nu}|_K$ is equal to the multiplicity of $\sigma$ in $q|_M$. Thus

$$\pi^{\sigma,\nu}|_K = \bigoplus_{\substack{q\in\hat{K}\\ \sigma\prec q}} q.$$

Hence, the multiplicity of every $q=(m_1,\ldots,m_k)\in\hat{K}$ in the elementary representation $\pi^{\sigma,\nu}$ is either 1 or 0 and the $K-$spectrum $\Gamma(\pi^{\sigma,\nu})$ consists of all $q=(m_1,\ldots,m_k)\in\hat{K}\cap(n_1+\mathbb{Z})^k$ such that

$$m_1 \geq n_1 \geq m_2 \geq n_2 \geq \cdots \geq m_{k-1} \geq n_{k-1} \geq \begin{cases} |m_k| & \text{if}\, n=2k,\\ m_k \geq |n_k| & \text{if}\, n=2k+1.\end{cases}$$

# 3 Representations of $\mathrm{Spin}(2k,1)$

In this section we first write down in our notation the known results on elementary representations and its irreducible subquotients for the groups $\mathrm{Spin}(2k,1)$ (see [1], [2], [3], [4], [5], [6], [9], [10]). For $\sigma=(n_1,\ldots,n_{k-1})$ in $\hat{M}\subseteq\mathbb{R}^{k-1}=i\mathfrak{d}_0^*$ and for $\nu\in\mathbb{C}=\mathfrak{a}^*$ the elementary representation $\pi^{\sigma,\nu}$ is irreducible if and only if either $\nu\notin\frac{1}{2}+n_1+\mathbb{Z}$ or

$$\nu \in \left\{\pm\left(n_{k-1}+\tfrac{1}{2}\right), \pm\left(n_{k-2}+\tfrac{3}{2}\right), \ldots, \pm\left(n_2+k-\tfrac{5}{2}\right), \pm\left(n_1+k-\tfrac{3}{2}\right)\right\}.$$

If $\pi^{\sigma,\nu}$ is reducible it has either two or three irreducible subquotients. If it has two, we will denote them by $\tau^{\sigma,\nu}$ and $\omega^{\sigma,\nu}$; an exception is the case of nonintegral $n_j$ and $\nu = 0$, when we denote them by $\omega^{\sigma,0,\pm}$. If $\pi^{\sigma,\nu}$ has three irreducible subquotients, we will denote them by $\tau^{\sigma,\nu}$ and $\omega^{\sigma,\nu,\pm}$. Their $K-$spectra are as follows:

$(a_1)$ $n_j \in \mathbb{Z}_+$ and $\nu \in \left\{\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots, \pm\left(n_{k-1} - \frac{1}{2}\right)\right\}$ (this is possible only if $n_{k-1} \geq 1$) :

$$\Gamma(\tau^{\sigma,\nu}) : \qquad m_1 \geq n_1 \geq \cdots \geq m_{k-1} \geq n_{k-1},\ |m_k| \leq |\nu| - \tfrac{1}{2};$$

$$\Gamma(\omega^{\sigma,\nu,\pm}) : \quad m_1 \geq n_1 \geq \cdots \geq m_{k-1} \geq n_{k-1} \geq \pm m_k \geq |\nu| + \tfrac{1}{2}.$$

$(a_2)$ $n_j \in \left(\frac{1}{2} + \mathbb{Z}_+\right)$ and $\nu \in \left\{\pm 1, \ldots, \pm\left(n_{k-1} - \frac{1}{2}\right)\right\}$ (this is possible only if $n_{k-1} \geq \frac{3}{2}$) :

$$\Gamma(\tau^{\sigma,\nu}) : \qquad m_1 \geq n_1 \geq \cdots \geq m_{k-1} \geq n_{k-1},\ |m_k| \leq |\nu| - \tfrac{1}{2};$$

$$\Gamma(\omega^{\sigma,\nu,\pm}) : \quad m_1 \geq n_1 \geq \cdots \geq m_{k-1} \geq n_{k-1} \geq \pm m_k \geq |\nu| + \tfrac{1}{2}.$$

$(a_3)$ $n_j \in \left(\frac{1}{2} + \mathbb{Z}_+\right)$ and $\nu = 0$ :

$$\Gamma(\omega^{\sigma,0,\pm}) : \quad m_1 \geq n_1 \geq \cdots \geq m_{k-1} \geq n_{k-1} \geq \pm m_k \geq \tfrac{1}{2}.$$

$(b)$ If $n_{j-1} > n_j$ for some $j \in \{2, \ldots, k-1\}$ and if

$$\nu \in \left\{\pm\left(n_j + k - j + \tfrac{1}{2}\right), \pm\left(n_j + k - j + \tfrac{3}{2}\right), \ldots, \right.$$
$$\left. \pm\left(n_{j-1} + k - j - \tfrac{1}{2}\right)\right\},$$

then:

$$\Gamma(\tau^{\sigma,\nu}) : \qquad m_1 \geq n_1 \geq \cdots \geq m_{j-1} \geq n_{j-1},$$
$$|\nu| - k + j - \tfrac{1}{2} \geq m_j \geq n_j \geq \cdots \geq n_{k-1} \geq |m_k|;$$

$$\Gamma(\omega^{\sigma,\nu}) : \qquad m_1 \geq n_1 \geq \cdots \geq n_{j-1} \geq m_j \geq |\nu| - k + j + \tfrac{1}{2},$$
$$n_j \geq m_{j+1} \geq \cdots \geq n_{k-1} \geq |m_k|.$$

$(c)$ $\nu \in \left\{\pm\left(n_1 + k - \frac{1}{2}\right), \pm\left(n_1 + k + \frac{1}{2}\right), \pm\left(n_1 + k + \frac{3}{2}\right), \ldots\right\}$ :

$$\Gamma(\tau^{\sigma,\nu}) : \quad |\nu| - k + \tfrac{1}{2} \geq m_1 \geq n_1 \geq \cdots \geq m_{k-1} \geq n_{k-1} \geq |m_k|;$$

$$\Gamma(\omega^{\sigma,\nu}) : \quad m_1 \geq |\nu| - k + \tfrac{3}{2},\ n_1 \geq m_2 \geq \cdots$$
$$\cdots \geq m_{k-1} \geq n_{k-1} \geq |m_k|.$$

Irreducible elementary representation $\pi^{\sigma,\nu}$ is unitary if and only if either $\nu \in i\mathbb{R}$ (so called **unitary principal series**) or $\nu \in \langle -\nu(\sigma), \nu(\sigma)\rangle$, where

$$\nu(\sigma) = \min\{\nu \geq 0;\ \pi^{\sigma,\nu} \text{ is reducible}\}$$

(so caled **complementary series**). Notice that for nonintegral $n_j$'s $\pi^{\sigma,0}$ is reducible, thus $\nu(\sigma) = 0$ and the complementary series is empty. In the case of integral $n_j$'s we have the following possibilities:

($a$) If $n_{k-1} \geq 1$, then $\nu(\sigma) = \frac{1}{2}$. The reducible elementary representation $\pi^{\sigma,\frac{1}{2}}$ is of the type ($a1$).

($b$) If $n_{k-1} = 0$ and $n_1 \geq 1$, let $j \in \{2, \dots, k-1\}$ be such that $n_{k-1} = \cdots = n_j = 0 < n_{j-1}$. Then $\nu(\sigma) = k - j + \frac{1}{2}$. The reducible elementary representation $\pi^{\sigma,k-j+\frac{1}{2}}$ is of the type ($bj$).

($c$) If $\sigma$ is trivial, i.e. $n_1 = \cdots = n_{k-1} = 0$, then $\nu(\sigma) = k - \frac{1}{2}$. The reducible elementary representation $\pi^{\sigma,k-\frac{1}{2}}$ is of the type ($c$).

Among irreducible subquotients of reducible elementary representations the unitary ones are $\omega^{\sigma,\nu,\pm}$, $\tau^{\sigma,\nu(\sigma)}$ and $\omega^{\sigma,\nu(\sigma)}$.

The infinitesimal character of $\pi^{\sigma,\nu}$ (and of its irreducible subquotients) is $\chi_{\Lambda(\sigma,\nu)}$, where $\Lambda(\sigma,\nu) \in \mathfrak{h}^*$ is given by

$$\Lambda(\sigma,\nu) = \left(n_1 + k - \tfrac{3}{2}, n_2 + k - \tfrac{5}{2}, \dots, n_{k-1} + \tfrac{1}{2}, \nu\right).$$

As we pointed out, if $\mathfrak{t}^*$ is identified with $\mathbb{C}^k$ through the basis $(H_1, \dots, H_k)$ of $\mathfrak{t}$, the same parameters determine this infinitesimal character with respect to the Harish−Chandra isomorphism $\mathfrak{Z}(\mathfrak{g}) \longrightarrow \mathcal{P}(\mathfrak{t}^*)^{W(\mathfrak{g},\mathfrak{t})}$.

The $W_K$−chamber in $\mathbb{R}^k = i\mathfrak{t}_0^*$ corresponding to chosen positive roots $\Delta_K^+$ is

$$C = \{\lambda \in \mathbb{R}^k;\ \lambda_1 > \lambda_2 > \ldots > \lambda_{k-1} > |\lambda_k| > 0\}.$$

The set $\mathcal{D}$ of $W$−chambers contained in $C$ consists of two elements:

$$D_\pm = \{\lambda \in \mathbb{R}^k;\ \lambda_1 > \lambda_2 > \cdots > \lambda_{k-1} > \pm\lambda_k > 0\}.$$

The closure $\overline{D}_+$ is fundamental domain for the action of $W$ on $\mathbb{R}^k$, i.e. each $W$−orbit in $\mathbb{R}^k$ intersects with $\overline{D}_+$ in one point. We saw that the reducibility criteria imply that $\Lambda(\sigma,\nu) \in \mathbb{R}^k$ whenever $\pi^{\sigma,\nu}$ is reducible. We denote by $\lambda(\sigma,\nu)$ the unique point in the intersection of $W\Lambda(\sigma,\nu)$ with $\overline{D}_+$. In the following theorem (proved with all details in [7]) we can suppose without loss of generality that $\nu \geq 0$, since $\pi^{\sigma,\nu}$ and $\pi^{\sigma,-\nu}$ have the same irreducible subquotients.

**Theorem 1.** *(i) $\pi^{\sigma,\nu}$ is reducible if and only if its infinitesimal character is $\chi_\lambda$ for some $\lambda \in \Lambda$, where*

$$\Lambda = \left\{ \lambda \in \mathbb{Z}_+^k \cup \left(\tfrac{1}{2} + \mathbb{Z}_+\right)^k ; \ \lambda_1 > \lambda_2 > \cdots > \lambda_{k-1} > \lambda_k \geq 0 \right\}.$$

*We write $\Lambda$ as the disjoint union $\Lambda^* \cup \Lambda^0$, where*

$$\Lambda^* = \left\{ \lambda \in \mathbb{Z}_+^k \cup \left(\tfrac{1}{2} + \mathbb{Z}_+\right)^k ; \ \lambda_1 > \lambda_2 > \cdots > \lambda_{k-1} > \lambda_k > 0 \right\},$$

$$\Lambda^0 = \left\{ \lambda \in \mathbb{Z}_+^k ; \ \lambda_1 > \lambda_2 > \cdots > \lambda_{k-1} > 0, \ \lambda_k = 0 \right\}.$$

*(ii) For $\lambda \in \Lambda^*$ there exist $k$ ordered pairs $(\sigma, \nu)$, $\sigma \in \hat{M}$, $\nu \geq 0$, such that $\chi_\lambda$ is the infinitesimal character of $\pi^{\sigma,\nu}$. These ordered pairs are $(\sigma_j, \nu_j)$, $1 \leq j \leq k$, where $\nu_j = \lambda_j$ and:*

$$\begin{aligned}
\sigma_1 &= \left(\lambda_2 - k + \tfrac{3}{2}, \ldots, \lambda_{s+1} - k + s + \tfrac{1}{2}, \ldots, \lambda_k - \tfrac{1}{2}\right), \\
\sigma_j &= \left(\lambda_1 - k + \tfrac{3}{2}, \ldots, \lambda_{j-1} - k + j - \tfrac{1}{2}, \lambda_{j+1} - k + j + \tfrac{1}{2}, \ldots, \lambda_k - \tfrac{1}{2}\right), \\
& \quad 2 \leq j \leq k-1, \\
\sigma_k &= \left(\lambda_1 - k + \tfrac{3}{2}, \lambda_2 - k + \tfrac{5}{2}, \ldots, \lambda_{k-1} - \tfrac{1}{2}\right).
\end{aligned}$$

*(iii) For $\lambda \in \Lambda^0$, the ordered pair $(\sigma, \nu)$, $\sigma \in \hat{M}$, $\nu \in \mathbb{R}$, such that $\chi_\lambda$ is the infinitesimal character of $\pi^{\sigma,\nu}$, is unique:*

$$\sigma = \left(\lambda_1 - k + \tfrac{3}{2}, \lambda_2 - k + \tfrac{5}{2}, \ldots, \lambda_{k-1} - \tfrac{1}{2}\right), \ \nu = 0.$$

Fix now $\lambda \in \Lambda^*$. There are altogether $k+2$ mutually infinitesimally inequivalent irreducible subquotients of the reducible elementary representations $\pi^{\sigma_1,\nu_1}, \ldots, \pi^{\sigma_k,\lambda_k}$ which we denote by $\tau_1^\lambda, \ldots, \tau_k^\lambda, \omega_+^\lambda, \omega_-^\lambda$ : $\tau_j^\lambda = \tau^{\sigma_j,\nu_j}$, $\omega_\pm^\lambda = \omega^{\sigma_k,\nu_k,\pm}$. Note that $\omega^{\sigma_j,\nu_j} \cong \tau_{j+1}^\lambda$ for $1 \leq j \leq k-1$.

The $K-$spectra of these irreducible representations consist of all $q = (m_1, \ldots, m_k)$ in $\hat{K} \cap \left(\lambda_1 + \frac{1}{2} + \mathbb{Z}\right)^k$ that satisfy:

$$\begin{aligned}
\Gamma(\tau_1^\lambda): &\quad \lambda_1 - k + \tfrac{1}{2} \geq m_1 \geq \lambda_2 - k + \tfrac{3}{2} \geq \cdots \geq m_{k-1} \geq \lambda_k - \tfrac{1}{2} \geq |m_k|, \\
&\quad \vdots \\
\Gamma(\tau_j^\lambda): &\quad m_1 \geq \lambda_1 - k + \tfrac{3}{2} \geq m_2 \geq \cdots \geq m_{j-1} \geq \lambda_{j-1} - k + j - \tfrac{1}{2}, \\
&\quad \lambda_j - k + j - \tfrac{1}{2} \geq m_j \geq \cdots \geq m_{k-1} \geq \lambda_k - \tfrac{1}{2} \geq |m_k|, \\
&\quad \vdots \\
\Gamma(\tau_k^\lambda): &\quad m_1 \geq \lambda_1 - k + \tfrac{3}{2} \geq m_2 \geq \lambda_2 - k + \tfrac{5}{2} \geq \cdots m_{k-1} \geq \lambda_{k-1} - \tfrac{1}{2}, \\
&\quad \lambda_k - \tfrac{1}{2} \geq |m_k|, \\
\Gamma(\omega_\pm^\lambda): &\quad m_1 \geq \lambda_1 - k + \tfrac{3}{2} \geq \cdots \geq m_{k-1} \geq \lambda_{k-1} - \tfrac{1}{2} \geq \pm m_k \geq \lambda_k + \tfrac{1}{2}.
\end{aligned}$$

It is obvious that each of these representations $\pi$ has one $D_+-$corner and one $D_--$corner; we denote them by $q_\pm(\pi)$. The list is:

$$\begin{aligned}
q_\pm(\tau_1^\lambda) &= \left(\lambda_2-k+\tfrac{3}{2},\ldots,\lambda_{k-1}-\tfrac{3}{2},\lambda_k-\tfrac{1}{2},\mp(\lambda_k-\tfrac{1}{2})\right),\\
q_\pm(\tau_j^\lambda) &= \left(\lambda_1-k+\tfrac{3}{2},\ldots,\lambda_{j-1}-k+j-\tfrac{1}{2},\right.\\
&\qquad \left.\lambda_{j+1}-k+j+\tfrac{1}{2},\ldots,\lambda_k-\tfrac{1}{2},\mp(\lambda_k-\tfrac{1}{2})\right),\\
q_\pm(\tau_k^\lambda) &= \left(\lambda_1-k+\tfrac{3}{2},\lambda_2-k+\tfrac{5}{2},\ldots,\lambda_{k-1}-\tfrac{1}{2},\mp(\lambda_k-\tfrac{1}{2})\right),\\
q_\pm(\omega_\pm^\lambda) &= \left(\lambda_1-k+\tfrac{3}{2},\lambda_2-k+\tfrac{5}{2},\ldots,\lambda_{k-1}-\tfrac{1}{2},\pm(\lambda_k+\tfrac{1}{2})\right),\\
q_\pm(\omega_\mp^\lambda) &= \left(\lambda_1-k+\tfrac{3}{2},\lambda_2-k+\tfrac{5}{2},\ldots,\lambda_{k-1}-\tfrac{1}{2},\mp(\lambda_{k-1}-\tfrac{1}{2})\right).
\end{aligned}$$

We check directly that among them the fundamental ones are $q_\pm(\tau_k^\lambda)$ and $q_\pm(\omega_\pm^\lambda)$ while the others $q_\pm(\tau_j^\lambda)$, $j<k$, $q_\pm(\omega_\mp^\lambda)$, are not fundamental.

Notice that finite dimensional $\tau_1^\lambda$ is not unitary and $q_+(\tau_1^\lambda)\neq q_-(\tau_1^\lambda)$ unless it is the trivial $1-$dimensional representation $(\lambda=\left(k-\frac{1}{2},k-\frac{3}{2},\ldots,\frac{1}{2}\right))$ when $q_+(\tau_1^\lambda)=q_-(\tau_1^\lambda)=(0,\ldots,0)$. Next, $\tau_j^\lambda$ for $2\leq j\leq k$ is not unitary and $q_+(\tau_j^\lambda)\neq q_-(t_j^\lambda)$. Finally, $\omega_+^\lambda$ and $\omega_-^\lambda$ are unitary (these are the discrete series representations) and each of them has one fundamental corner, $q_+(\omega_+^\lambda)$ and $q_-(\omega_-^\lambda)$; the other two $q_-(\omega_+^\lambda)$ and $q_+(\omega_-^\lambda)$ are not fundamental.

We consider now the case $\lambda\in\Lambda^0$. The elementary representation $\pi^{\sigma,0}$ is unitary and it is direct sum of two unitary irreducible representations $\omega_+^\lambda$ and $\omega_-^\lambda$. Their $K-$spectra consist of all $q=(m_1,\ldots,m_k)\in\hat{K}\cap\left(\frac{1}{2}+\mathbb{Z}\right)^k$ that satisfy

$$\Gamma(\omega_\pm^\lambda):\quad m_1\geq\lambda_1-k+\tfrac{3}{2}\geq\cdots\geq m_{k-1}\geq\lambda_{k-1}-\tfrac{1}{2}\geq\pm m_k\geq\tfrac{1}{2}.$$

Again each of these representations have one $D_+-$corner and one $D_--$corner:

$$\begin{aligned}
q_\pm(\omega_\pm^\lambda) &= \left(\lambda_1-k+\tfrac{3}{2},\lambda_2-k+\tfrac{5}{2},\ldots,\lambda_{k-1}-\tfrac{1}{2},\pm\tfrac{1}{2}\right),\\
q_\pm(\omega_\mp^\lambda) &= \left(\lambda_1-k+\tfrac{3}{2},\lambda_2-k+\tfrac{5}{2},\ldots,\lambda_{k-1}-\tfrac{1}{2},\mp(\lambda_{k-1}-\tfrac{1}{2})\right).
\end{aligned}$$

We find that again each of these unitary representation has one fundamental corner ($q_+(\omega_+^\lambda)$, resp. $q_-(\omega_-^\lambda)$), and the other corner is not fundamental.

To summarize, we see that $\pi\in\widehat{G}^0$ with exactly one fundamental corner is unitary; its fundamental corner we denote by $q(\pi)$. For all the others

$\pi \in \hat{G}^0$ one has $q_1(\pi) = q_2(\pi)$ and we denote by $q(\pi)$ this unique corner of $\pi$.

**Theorem 2.** *$\pi \mapsto q(\pi)$ is a bijection of $\hat{G}^0$ onto $\hat{K}$.*

Detailed proof can be find in [7].

Consider now minimal $K-$types in the sense of Vogan: we say that $q \in \hat{K}$ is a **minimal $K-$type** of the representation $\pi$ if $q \in \Gamma(\pi)$ and

$$\|q + 2\rho_K\| = \min\{\|q' + 2\rho_K\|;\ q' \in \Gamma(\pi)\}.$$

For $q \in \hat{K}$ we have

$$\|q+2\rho_K\|^2 = (m_1+2k-2)^2 + (m_2+2k-4)^2 + \cdots + (m_{k-1}+2)^2 + m_k^2$$

and so we find:

If $\lambda \in \Lambda \cap \left(\frac{1}{2}+\mathbb{Z}\right)^k$, i.e. $\lambda \in \Lambda^*$ and $\Gamma(\tau_j^\lambda) \subseteq \mathbb{Z}^k$, the representation $\tau_j^\lambda$ has unique minimal $K-$type which we denote by $q^V(\tau_j^\lambda)$ :

$$\begin{aligned}
q^V(\tau_1^\lambda) &= \left(\lambda_2 - k + \tfrac{3}{2}, \lambda_3 - k + \tfrac{5}{2}, \ldots, \lambda_k - \tfrac{1}{2}, 0\right), \\
q^V(\tau_j^\lambda) &= \left(\lambda_1 - k + \tfrac{3}{2}, \ldots, \lambda_{j-1} - k + j - \tfrac{1}{2},\right. \\
&\quad \left.\lambda_{j+1} - k + j + \tfrac{1}{2}, \ldots, \lambda_k - \tfrac{1}{2}, 0\right), \quad 2 \le j \le k-1, \\
q^V(\tau_k^\lambda) &= \left(\lambda_1 - k + \tfrac{3}{2}, \lambda_2 - k + \tfrac{5}{2}, \ldots, \lambda_{k-1} - \tfrac{1}{2}, 0\right).
\end{aligned}$$

If $\lambda \in \Lambda \cap \mathbb{Z}^k$, i.e. $\Gamma(\tau_j^\lambda) \subseteq \left(\frac{1}{2}+\mathbb{Z}\right)^k$, the representation $\tau_j^\lambda$ has two minimal $K-$types $q_+^V(\tau_j^\lambda)$ and $q_-^V(\tau_j^\lambda)$ :

$$\begin{aligned}
q_\pm^V(\tau_1^\lambda) &= \left(\lambda_2 - k + \tfrac{3}{2}, \lambda_3 - k + \tfrac{5}{2}, \ldots, \lambda_k - \tfrac{1}{2}, \pm\tfrac{1}{2}\right), \\
q_\pm^V(\tau_j^\lambda) &= \left(\lambda_1 - k + \tfrac{3}{2}, \ldots, \lambda_{j-1} - k + j - \tfrac{1}{2},\right. \\
&\quad \left.\lambda_{j+1} - k + j + \tfrac{1}{2}, \ldots, \lambda_k - \tfrac{1}{2}, \pm\tfrac{1}{2}\right), \quad 2 \le j \le k-1, \\
q_\pm^V(\tau_k^\lambda) &= \left(\lambda_1 - k + \tfrac{3}{2}, \lambda_2 - k + \tfrac{5}{2}, \ldots, \lambda_{k-1} - \tfrac{1}{2}, \pm\tfrac{1}{2}\right).
\end{aligned}$$

Finally, for every $\lambda \in \Lambda$ the representation $\omega_\pm^\lambda$ has unique minimal $K-$type:

$$q^V(\omega_\pm^\lambda) = \left(\lambda_1 - k + \tfrac{3}{2}, \lambda_2 - k + \tfrac{5}{2}, \ldots, \lambda_{k-1} - \tfrac{1}{2}, \pm(\lambda_k + \tfrac{1}{2})\right).$$

So we see that if $\pi \in \widehat{G}^0$ has two minimal $K-$types it is not unitary. Further, every $\pi \in \hat{G}^0$ has unique minimal $K-$type $q^V(\pi)$ and it coincides with $q(\pi)$. But there exist nonunitary representations in $\widehat{G}^0$ that have unique minimal $K-$type: this property have all $\tau_j^\lambda$ for $\lambda \in \Lambda \cap \left(\frac{1}{2}+\mathbb{Z}_+\right)^k$ that are not subquotients of the ends of complementary series. In other words, unitarity of a representation $\pi \in \widehat{G}^0$ is not completely characterized by having unique minimal $K-$type.

# 4 Representations of Spin$(2k+1,1)$

For $\sigma = (n_1,\dots,n_k) \in \hat{M} \cap \mathbb{Z}^k$ and $\nu \in \mathbb{C}$ the elementary representation $\pi^{\sigma,\nu}$ is irreducible if and only if either $\nu \notin \mathbb{Z}$ or

$$\nu \in \{0,\pm 1,\dots,\pm|n_k|,\pm(n_{k-1}+1),\pm(n_{k-2}+2),\dots,\pm(n_1+k-1)\}.$$

For $\sigma \in \hat{M} \cap \left(\frac{1}{2}+\mathbb{Z}\right)^k$ and $\nu \in \mathbb{C}$ the representation $\pi^{\sigma,\nu}$ is irreducible if and only if either $\nu \notin \left(\frac{1}{2}+\mathbb{Z}\right)$ or

$$\nu \in \left\{\pm\tfrac{1}{2},\dots,\pm|n_k|,\pm(n_{k-1}+1),\pm(n_{k-2}+2),\dots,\pm(n_1+k-1)\right\}.$$

If the elementary representation $\pi^{\sigma,\nu}$ is reducible, it always has two irreducible subquotients which will be denoted by $\tau^{\sigma,\nu}$ and $\omega^{\sigma,\nu}$. The $K-$spectra of these representations consist of all $q = (m_1,\dots,m_k) \in \hat{K} \cap (n_1+\mathbb{Z})^k$ that satisfy:

*(i)* If $n_{k-1} > |n_k|$ and $\nu \in \{\pm(|n_k|+1),\pm(|n_k|+2),\dots,\pm n_{k-1}\}$ :

$$\Gamma(\tau^{\sigma,\nu}):\ m_1 \geq n_1 \geq \cdots \geq m_{k-1} \geq n_{k-1} \text{ and } |\nu|-1 \geq m_k \geq |n_k|,$$

$$\Gamma(\omega^{\sigma,\nu}):\ m_1 \geq n_1 \geq \cdots \geq m_{k-1} \geq n_{k-1} \geq m_k \geq |\nu|.$$

*(ii)* If $n_{j-1} > n_j$ for some $j \in \{2,\dots,k-1\}$ and

$\nu \in \{\pm(n_j+k-j+1),\pm(n_j+k-j+2),\dots,\pm(n_{j-1}+k-j)\}$ :

$$\Gamma(\tau^{\sigma,\nu}):\ m_1 \geq n_1 \geq \cdots \geq m_{j-1} \geq n_{j-1} \text{ and } |\nu|-k+j-1 \geq m_j \geq n_j \geq \cdots \geq m_k \geq |n_k|,$$

$$\Gamma(\omega^{\sigma,\nu}):\ m_1 \geq n_1 \geq \cdots \geq m_{j-1} \geq n_{j-1} \geq m_j \geq |\nu|-k+j \text{ and } n_j \geq m_{j+1} \geq \cdots \geq m_k \geq |n_k|.$$

*(iii)* If $\nu \in \{\pm(n_1+k),\pm(n_1+k+1),\dots\}$ :

$$\Gamma(\tau^{\sigma,\nu}):\ |\nu|-k \geq m_1 \geq n_1 \geq \cdots \geq m_k \geq |n_k|,$$

$$\Gamma(\omega^{\sigma,\nu}):\ m_1 \geq |\nu|-k+1 \text{ and } n_1 \geq m_2 \geq n_2 \geq \cdots \geq m_k \geq |n_k|.$$

Similarly to the case of even $n = 2k$ we now write down the infinitesimal characters of reducible elementary representations $\pi^{\sigma,\nu}$ (and so of its irreducible subquotients $\tau^{\sigma,\nu}$ and $\omega^{\sigma,\nu}$ too). We know that the infinitesimal character of $\pi^{\sigma,\nu}$ is $\chi_{\Lambda(\sigma,\nu)}$, where

$$\Lambda(\sigma,\nu) = (n_1 + k - 1, n_2 + k - 2, \ldots, n_{k-1} + 1, n_k, \nu).$$

Since $\nu \in \frac{1}{2}\mathbb{Z} \subset \mathbb{R} = \mathfrak{a}_0^*$ we have $\Lambda(\sigma,\nu) \in i\mathfrak{t}_0^* \oplus \mathfrak{a}_0^* = \mathbb{R}^{k+1}$. We choose positive Weyl chamber in $\mathbb{R}^{k+1}$

$$D = \{\lambda \in \mathbb{R}^{k+1};\ \lambda_1 > \lambda_2 > \cdots > \lambda_k > |\lambda_{k+1}| > 0\}$$

and again denote by $\lambda(\sigma,\nu)$ the unique point of $W\Lambda(\sigma,\nu) \cap \overline{D}$. We now write down $\lambda(\sigma,\nu)$ for all reducible elementary representations $\pi^{\sigma,\nu}$. In the following for $\sigma = (n_1,\ldots,n_k) \in \hat{M}$ we write $-\sigma$ for its contragredient class in $\hat{M} : -\sigma = (n_1,\ldots,n_{k-1},-n_k)$. Without loss of generality we can suppose that $\nu \geq 0$ because $\pi^{\sigma,\nu}$ and $\pi^{-\sigma,-\nu}$ have equivalent irreducible subquotients and because $\Lambda(\sigma,\nu)$ is $W-$conjugated with $\Lambda(-\sigma,-\nu)$ : multiplying the last two coordinates by $-1$.

If $n_{k-1} > |n_k|$ and $\nu \in \{|n_k| + 1, |n_k| + 2, \ldots, n_{k-1}\}$,

$$\lambda(\sigma,\nu) = (n_1 + k - 1, n_2 + k - 2, \ldots, n_{k-1} + 1, \nu, n_k).$$

If $2 \leq j \leq k-1$, $n_{j-1} > n_j$ and $\nu \in \{n_j + k - j + 1, \ldots, n_{j-1} + k - j\}$,

$$\lambda(\sigma,\nu) = (n_1 + k - 1, \ldots, n_{j-1} + k - j + 1,\ \nu\ , n_j + k - j, \ldots, n_{k-1} + 1, n_k).$$

If $\nu \in \{n_1 + k, n_1 + k + 1, \ldots\}$,

$$\lambda(\sigma,\nu) = (\nu, n_1 + k - 1, \ldots, n_{k-1} + 1, n_k).$$

Similarly to the case of even $n$ we see that now every reducible elementary representation has infinitesimal character $\chi_\lambda$ with $\lambda \in \Lambda$, where

$$\Lambda = \left\{\lambda \in \mathbb{Z}^{k+1} \cup \left(\tfrac{1}{2} + \mathbb{Z}\right)^{k+1};\ \lambda_1 > \lambda_2 > \cdots > \lambda_k > |\lambda_{k+1}|\right\}.$$

We again write $\Lambda$ as the disjoint union $\Lambda = \Lambda^* \cup \Lambda^0$, where

$$\begin{aligned}
\Lambda^* &= \left\{\lambda \in \mathbb{Z}^{k+1} \cup \left(\tfrac{1}{2} + \mathbb{Z}\right)^{k+1};\ \lambda_1 > \lambda_2 > \cdots > \lambda_k > |\lambda_{k+1}| > 0\right\}, \\
\Lambda^0 &= \{\lambda \in \mathbb{Z}_+^{k+1};\ \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0,\ \lambda_{k+1} = 0\}.
\end{aligned}$$

As shown in [7] we have

**Theorem 3.** *(i) For every $\lambda \in \Lambda^*$ there exist $k+1$ ordered pairs $(\sigma,\nu)$, $\sigma = (n_1,\dots,n_k) \in \hat{M}$, $\nu \geq 0$, such that $\chi_\lambda$ is the infinitesimal character of $\pi^{\sigma,\nu}$. These are $(\sigma_j,\nu_j)$, where $\nu_j = \lambda_j$ for $1 \leq j \leq k$, $\nu_{k+1} = |\lambda_{k+1}|$ and*

$$\begin{aligned}
\sigma_1 &= (\lambda_2 - k + 1, \lambda_3 - k + 2, \dots, \lambda_k - 1, \lambda_{k+1}),\\
\sigma_j &= (\lambda_1 - k + 1, \dots, \lambda_{j-1} - k + j - 1, \lambda_{j+1} - k + j, \dots, \lambda_k - 1, \lambda_{k+1}),\\
&\qquad 2 \leq j \leq k-1,\\
\sigma_k &= (\lambda_1 - k + 1, \lambda_2 - k + 2, \dots, \lambda_{k-1} - 1, \lambda_{k+1}),\\
\sigma_{k+1} &= \begin{cases} (\lambda_1 - k + 1, \lambda_2 - k + 2, \dots, \lambda_{k-1} - 1, \lambda_k) & \text{if } \lambda_{k+1} > 0,\\ (\lambda_1 - k + 1, \lambda_2 - k + 2, \dots, \lambda_{k-1} - 1, -\lambda_k) & \text{if } \lambda_{k+1} < 0, \end{cases}
\end{aligned}$$

*$\pi^{\sigma_j,\nu_j}$, $1 \leq j \leq k$, are reducible, while $\pi^{\sigma_{k+1},\nu_{k+1}}$ is irreducible.*

*(ii) For $\lambda \in \Lambda^0$ there exist $k+2$ ordered pairs $(\sigma,\nu)$, $\sigma = (n_1,\dots,n_k) \in \hat{M}$, $\nu \geq 0$, such that $\chi_\lambda$ is the infinitesimal character of $\pi^{\sigma,\nu}$. These are the $(\sigma_j,\nu_j)$, where $\nu_j = \lambda_j$ for $1 \leq j \leq k$, $\nu_{k+1} = \nu_{k+2} = 0$ and*

$$\begin{aligned}
\sigma_1 &= (\lambda_2 - k + 1, \lambda_3 - k + 2, \dots, \lambda_k - 1, 0),\\
\sigma_j &= (\lambda_1 - k + 1, \dots, \lambda_{j-1} - k + j - 1, \lambda_{j+1} - k + j, \dots, \lambda_k - 1, 0),\\
&\qquad 2 \leq j \leq k-1,\\
\sigma_k &= (\lambda_1 - k + 1, \lambda_2 - k + 2, \dots, \lambda_{k-1} - 1, 0),\\
\sigma_{k+1} &= (\lambda_1 - k + 1, \lambda_2 - k + 2, \dots, \lambda_{k-1} - 1, \lambda_k),\\
\sigma_{k+2} &= (\lambda_1 - k + 1, \lambda_2 - k + 2, \dots, \lambda_{k-1} - 1, -\lambda_k).
\end{aligned}$$

*$\pi^{\sigma_j,\nu_j}$, $1 \leq j \leq k$, are reducible, while $\pi^{\sigma_{k+1},0}$ and $\pi^{\sigma_{k+2},0}$ are irreducible.*

We note that in fact the representations $\pi^{\sigma_{k+1},0}$ and $\pi^{\sigma_{k+2},0}$ are equivalent, but this is unimportant for studying and parametrizing $\widehat{G}^0$ and $\hat{G}^0$.

Fix $\lambda \in \Lambda$. By Theorem 3. there exist $k$ ordered pairs $(\sigma,\nu)$, $\sigma \in \hat{M}$, $\nu \geq 0$, with reducible $\pi^{\sigma,\nu}$ having $\chi_\lambda$ as the infinitesimal character. There are $k+1$ mutually inequivalent irreducible subquotients of these elementary representations; we denote them $\tau_1^\lambda,\dots,\tau_k^\lambda,\omega^\lambda : \tau_j^\lambda = \tau^{\sigma_j,\nu_j}$, $1 \leq j \leq k$, $\omega^\lambda = \omega^{\sigma_k,\nu_k}$. Note that $\omega^{\sigma_j,\nu_j} \cong \tau_{j+1}^\lambda$ for $1 \leq j \leq k-1$. Their

$K-$spectra consist of all $q = (m_1, \ldots, m_k) \in \hat{K} \cap (n_1 + \mathbb{Z})^k$ satisfying:

$\Gamma(\tau_1^\lambda):\quad \lambda_1 - k \geq m_1 \geq \lambda_2 - k + 1 \geq m_2 \geq \cdots \geq \lambda_k - 1 \geq m_k \geq |\lambda_{k+1}|.$

$\Gamma(\tau_j^\lambda):\quad m_1 \geq \lambda_1 - k + 1 \geq \cdots \geq m_{j-1} \geq \lambda_{j-1} - k + j - 1$ and
$\lambda_j - k + j - 1 \geq m_j \geq \cdots \geq \lambda_k - 1 \geq m_k \geq |\lambda_{k+1}|$
for $2 \leq j \leq k$.

$\Gamma(\omega^\lambda):\quad m_1 \geq \lambda_1 - k + 1 \geq \cdots \geq m_{k-1} \geq \lambda_{k-1} - 1 \geq m_k \geq \lambda_k.$

The definitions of corners and fundamental corners do not have sense when $\operatorname{rank}\mathfrak{k} < \operatorname{rank}\mathfrak{g}$. Consider the Vogan's minimal $K-$types. Note that now

$$\|q + 2\rho_K\|^2 = (m_1 + 2k - 1)^2 + (m_2 + 2k - 3)^2 + \cdots + (m_k + 1)^2,$$

so every $\pi \in \widehat{G}^0$ has unique minimal $K-$type that will be denoted by $q^V(\pi)$ : this is the element $(m_1, \ldots, m_k) \in \Gamma(\pi)$ whose every coordinate $m_j$ is the smallest possible.

**Theorem 4.** *The map $\pi \mapsto q^V(\pi)$ is a surjection of $\widehat{G}^0$ onto $\hat{K}$. More precisely, for every $q = (m_1, \ldots, m_k) \in \hat{K}$ :*

*(a) There exist infinitely many $\lambda$'s in $\Lambda$ such that $q^V(\tau_1^\lambda) = q$.*

*(b) Let $j \in \{2, \ldots, k\}$. The number of $\lambda$'s in $\Lambda$ such that $q^V(\tau_j^\lambda) = q$ is:*

$$\begin{array}{ll} 0 & \text{if } m_{j-1} = m_j, \\ m_{j-1} - m_j & \text{if } m_{j-1} > m_j \text{ and } m_k = 0, \\ 2(m_{j-1} - m_j) & \text{if } m_{j-1} > m_j \text{ and } m_k > 0. \end{array}$$

*(c) The number of $\lambda$'s in $\Lambda$ such that $q^V(\omega^\lambda) = q$ is:*

$$\begin{array}{ll} 0 & \text{if } m_k < 1, \\ 1 & \text{if } m_k = 1, \\ 2\left\lfloor m_k - \frac{1}{2} \right\rfloor & \text{if } m_k > 1. \end{array}$$

*Proof.* $(a)$ These are all $\lambda \in \Lambda$ such that

$$\lambda_1 \in (m_1 + k + \mathbb{Z}_+),\ \ \lambda_j = m_{j-1} + k - j + 1\ \ 2 \leq j \leq k,\ \ \lambda_{k+1} = \pm m_k.$$

$(b)$ These are all $\lambda \in \Lambda$ such that $\lambda_s = m_s + k - s$ for $1 \leq s \leq j-1$, $\lambda_s = m_{s-1} + k - s + 1$ for $j + 1 \leq s \leq k$, $\lambda_{j-1} > \lambda_j > \lambda_{j+1}$ and $\lambda_{k+1} = \pm m_k$.

$(c)$ These are all $\lambda \in \Lambda$ such that

$$\lambda_s = m_s + k - s, \;\; 1 \leq s \leq k, \;\; |\lambda_{k+1}| < m_k.$$

$\square$

We now parametrize $\hat{G}^0$. A class in $\widehat{G}^0$ is unitary if and only if it is an irreducible subquotient of an end of complementary series. For $\sigma \in \hat{M}$ the complementary series is nonempty if and only if $\sigma$ is selfcontragredient, i.e. $\sigma = (n_1, \ldots, n_{k-1}, 0)$. In this case we set $\nu(\sigma) = \min\{\nu \geq 0;\ \pi^{\sigma,\nu}$ is reducible$\}$. From the necessary and sufficient conditions for reducibility of elementary representations we find:

$(i)$ If $n_1 = \cdots = n_{k-1} = 0$, i.e. if $\sigma = \sigma_0 = (0, \ldots, 0)$ is the trivial onedimensional representation of $M$, then $\nu(\sigma_0) = k$. In this case $\Gamma(\tau^{\sigma_0,k}) = \{(0, \ldots, 0)\}$ and $\Gamma(\omega^{\sigma_0,k}) = \{(s, 0, \ldots, 0);\ s \in \mathbb{N}\}$ and so $q^V(\tau^{\sigma_0,k}) = (0, \ldots, 0)$ and $q^V(\omega^{\sigma_0,k}) = (1, 0, \ldots, 0)$.

$(ii)$ If $n_1 > 0$, let $j \in \{2, \ldots, k\}$ be the smallest index such that $n_{j-1} > 0$. Then $\nu(\sigma) = k - j + 1$. The $K-$spectra of irreducible subquotients of $\pi^{\sigma,k-j+1}$ are

$$\begin{aligned}
\Gamma(\tau^{\sigma,k-j+1}): \quad & m_1 \geq n_1 \geq \cdots \geq m_{j-1} \geq n_{j-1} \\
& \text{and} \quad m_s = 0 \ \forall s \geq j, \\[1ex]
\Gamma(\omega^{\sigma,k-j+1}): \quad & m_1 \geq n_1 \geq \cdots \geq m_{j-1} \geq n_{j-1} \geq m_j \geq 1 \\
& \text{and} \quad m_s = 0 \ \forall s > j.
\end{aligned}$$

So we have

$$\begin{aligned}
q^V(\tau^{\sigma,k-j+1}) &= (n_1, \ldots, n_{j-1}, 0, \ldots, 0), \\
q^V(\omega^{\sigma,k-j+1}) &= (n_1, \ldots, n_{j-1}, 1, 0, \ldots, 0).
\end{aligned}$$

Thus, we have proved

**Theorem 5.** *The map $\pi \mapsto q^V(\pi)$ is a bijection of $\hat{G}^0$ onto*

$$\hat{K}_0 = \{q = (m_1, \ldots, m_k) \in \hat{K};\ m_k = 0\}.$$

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Domagoj Kovačević
University of Zagreb, Faculty of Electrical Engineering and Computing, Unska 3, 10 000 Zagreb, Croatia
*E-mail address:* `domagoj.kovacevic@fer.hr`

Hrvoje Kraljević
University of Zagreb, Faculty of Science, Department of Mathemamatics, Bijenička cesta 30, 10 000, Croatia

*E-mail address:* hrvoje.kraljevic@math.hr